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### **An Elementary Treatise on the Integral Calculus.**

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AN  
ELEMENTARY TREATISE  
ON THE  
INTEGRAL CALCULUS

FOUNDED ON THE  
METHOD OF RATES OR FLUXIONS

BY  
WILLIAM WOOLSEY JOHNSON  
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ANNAPOLIS MARYLAND

*REVISED EDITION.*

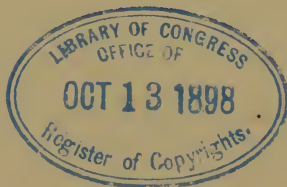
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# THE INTEGRAL CALCULUS.

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## CHAPTER I.

### ELEMENTARY METHODS OF INTEGRATION.

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#### I.

#### *Integrals.*

1. IN an important class of problems, the required quantities are magnitudes generated in given intervals of time with rates given in terms of the time  $t$ ; or else, being assumed to be so generated concurrently with some other independent variable, have rates expressible in terms of this independent variable and its rate.

For example, the velocity of a freely falling body is known to be expressed by the equation

$$v = gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which  $t$  is the number of seconds which have elapsed since the instant of rest, and  $g$  is a constant which has been determined experimentally. If  $s$  denotes the distance of the body



If we take this origin at the point occupied by the body when at rest, we shall have  $s = 0$  when  $t = 0$ , and therefore from equation (3)  $C = 0$ ; whence the equation becomes  $s = \frac{1}{2}gt^2$ .

### *The Differential of a Curvilinear Area.*

3. The area included between a curve, whose equation is given, the axis of  $x$  and two ordinates affords an instance of the second case mentioned in the first paragraph of Art. I; namely, that in which the rate of the generated quantity, although not given in terms of  $t$ , can be readily expressed by means of the assumed rate of some other independent variable.

Let  $BPD$  in Fig. 1 be the curve whose equation is supposed to be given in the form

$$y = f(x).$$

Supposing the variable ordinate  $PR$  to move from the position  $AB$  to the position  $CD$ , the required area  $ABDC$  is the final value of the variable area  $ABPR$ , denoted by

$A$ , which is generated by the motion of the ordinate. The rate at which the area  $A$  is generated can be expressed in terms of the rate of the independent variable  $x$ . The required and the assumed rates are denoted, respectively, by  $\frac{dA}{dt}$  and  $\frac{dx}{dt}$ ; and, to express the former in terms of the latter, it is necessary to express  $dA$  in terms of  $dx$ . Since  $x$  is an independent variable, we may assume  $dx$  to be constant; the rate at which  $A$  is generated is then a variable rate, because  $PR$  or  $y$  is of variable length, while  $R$  moves at a constant rate along the axis of  $x$ . Now  $dA$  is the increment which  $A$  would receive in the time

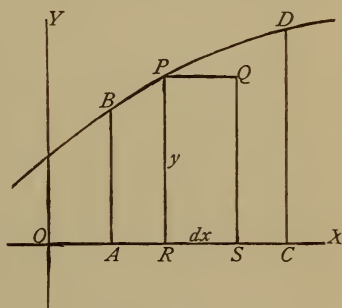


FIG. I.

$dt$ , were the rate of  $A$  to become constant (see Diff. Calc., Art. 17). If, now, at the instant when the ordinate passes the position  $PR$  in the figure, its length should become constant, the rate of the area would become constant, and the increment which would then be received in the time  $dt$ , namely, the rectangle  $PQSR$ , represents  $dA$ . Since the base  $RS$  of this rectangle is  $dx$ , we have

$$dA = ydx = f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, by the definition given in Art. 2,  $A$  is an integral, and is denoted by

$$A = \int f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

### *Definite Integrals.*

4. Equation (2) expresses that  $A$  is a function of  $x$ , whose differential is  $f(x)dx$ ; this function, like that considered in Art. 2, involves an undetermined constant. In fact, the expression  $\int f(x)dx$  is manifestly insufficient to represent precisely the area  $ABPR$ , because  $OA$ , the initial value of  $x$ , is not indicated. The indefinite character of this expression is removed by writing this value as a subscript to the integral sign; thus, denoting the initial value by  $a$ , we write

$$A = \int_a f(x)dx, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the subscript is *that value of  $x$  for which the integral has the value zero.*

If we denote the *final value* of  $x$  ( $OC$  in the figure) by  $b$ , the area  $ABDC$ , which is a particular value of  $A$ , is denoted by



writing this value of  $x$  at the top of the integral sign, thus,

$$ABDC = \int_a^b f(x) dx. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This last expression is called a *definite integral*, and  $a$  and  $b$  are called its *limits*. In contradistinction, the expression  $\int f(x) dx$  is called an *indefinite integral*.

5. As an application of the general expressions given in the last two articles, let the given curve be the parabola

$$y = x^2.$$

Equation (2) becomes in this case

$$A = \int x^2 dx.$$

Now, since  $\frac{1}{3}x^3$  is a function whose differential is  $x^2 dx$ , this equation gives

$$A = \int x^2 dx = \frac{1}{3}x^3 + C, \quad . \quad . \quad . \quad . \quad (I)$$

in which  $C$  is undetermined.

Now let us suppose the limiting ordinates of the required area to be those corresponding to  $x = 1$  and  $x = 3$ . The variable area of which we require a special value is now represented by  $\int_1^3 x^2 dx$ , which denotes that value of the indefinite integral which vanishes when  $x = 1$ . If we put  $x = 1$  in the general expression in equation (I), namely  $\frac{1}{3}x^3 + C$ , we have  $\frac{1}{3} + C$ ; hence if we subtract this quantity from the general expression, we shall have an expression which becomes zero when  $x = 1$ . We thus obtain

$$A = \int_1^3 x^2 dx = \frac{1}{3}x^3 - \frac{1}{3}.$$

Finally, putting, in this expression for the variable area,  $x = 3$ , we have for the required area

$$\int_1^3 x^2 dx = \frac{1}{3}3^3 - \frac{1}{3} = 8\frac{2}{3}.$$

6. It is evident that the definite integral obtained by this process is simply *the difference between the values of the indefinite integral at the upper and lower limits*. This difference may be expressed by attaching the limits to the symbol  $\int$  affixed to the value of the indefinite integral. Thus the process given in the preceding article is written thus,

$$\left[ \int_1^3 x^2 dx = \frac{1}{3}x^3 + C \right]_1^3 = 9 - \frac{1}{3} = 8\frac{2}{3}.$$

The essential part of this process is the determination of the indefinite integral or function whose differential is equal to the given expression. This is called the *integration* of the given differential expression.

### *Elementary Theorems.*

7. *A constant factor may be transferred from one side of the integral sign to the other.* In other words, if  $m$  is a constant and  $u$  a function of  $x$ ,

$$\int m u dx = m \int u dx.$$

Since each member of this equation involves an arbitrary constant, the equation only implies that the two members have the same differential. The differential of an integral is by definition the quantity under the integral sign. Now the second member is the product of a constant by a variable factor; hence its differential is  $m d \left[ \int u dx \right]$ , that is,  $m u dx$ , which is also the differential of the first member.

8. This theorem is useful not only in removing constant factors from under the integral sign, but also in introducing such factors when desired. Thus, given the integral

$$\int x^n dx;$$

recollecting that

$$d(x^{n+1}) = (n+1)x^n dx,$$

we introduce the constant factor  $n+1$  under the integral sign; thus,

$$\int x^n dx = \frac{1}{n+1} \int (n+1)x^n dx = \frac{1}{n+1} x^{n+1} + C.$$

9. If a differential expression be separated into parts, its integral is the sum of the integrals of the several parts. That is, if  $u, v, w, \dots$  are functions of  $x$ ,

$$\int (u + v + w + \dots) dx = \int u dx + \int v dx + \int w dx + \dots$$

For, since the differential of a sum is the sum of the differentials of the several parts, the differential of the second member is identical with that of the first member, and each member involves an arbitrary constant

Thus, for example,

$$\int (2 - \sqrt{x}) dx = \int 2 dx - \int x^{\frac{1}{2}} dx = 2x - \frac{2}{3} x^{\frac{3}{2}} + C,$$

the last term being integrated by means of the formula deduced in Art. 8.

### *Fundamental Integrals.*

10. The integrals whose values are given below are called the *fundamental integrals*. The constants of integration are generally omitted for convenience.

Formula (a) is given in two forms, the first of which is derived in Art. 8, while the second is simply the result of putting  $n = -m$ . It is to be noticed that this formula gives an indeterminate result when  $n = -1$ ; but in this case, formula (b) may be employed.\*

The remaining formulas are derived directly from the formulas for differentiation; except that  $(j')$ ,  $(k')$ ,  $(l')$ , and  $(m')$  are derived from  $(j)$ ,  $(k)$ ,  $(l)$ , and  $(m)$  by substituting  $\frac{x}{a}$  for  $x$ .

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad \int \frac{dx}{x^m} = -\frac{1}{(m-1)x^{m-1}} \quad \dots \quad (a)$$

$$\int \frac{dx}{x} = \log (\pm x) + . . . . . (b)$$

$$\int a^x dx = \frac{a^x}{\log a} \qquad \int \varepsilon^x dx = \varepsilon^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

[illegible]

$$\int \sin\theta d\theta = -\cos\theta . . . . . (e)$$

\* Applying formula (a) to the definite integral  $\int_a^b x^n dx$ , we have

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1},$$

which takes the form  $\frac{0}{0}$  when  $n = -1$ ; but, evaluating in the usual manner,

$$\lim_{n \rightarrow \infty} \frac{b^{n+1} - a^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{b^{n+1} \log b - a^{n+1} \log a}{1} = \log b - \log a;$$

a result identical with that obtained by employing formula (b).

† That sign is to be employed which makes the logarithm real. See Diff. Calc., Art. 43.

$$\int \frac{d\theta}{\cos^2 \theta} = \int \sec^2 \theta d\theta = \tan \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

$$\int \frac{d\theta}{\sin^2 \theta} = \int \operatorname{cosec}^2 \theta d\theta = -\cot \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

$$\int \frac{\sin \theta d\theta}{\cos^2 \theta} = \int \sec \theta \tan \theta d\theta = \sec \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

$$\int \frac{\cos \theta d\theta}{\sin^2 \theta} = \int \operatorname{cosec} \theta \cot \theta d\theta = -\operatorname{cosec} \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (j)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C = -\cos^{-1} \frac{x}{a} + C'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (j')$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C = -\frac{1}{a} \cot^{-1} \frac{x}{a} + C'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (k')$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = -\operatorname{cosec}^{-1} x + C'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (l)$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + C'. \quad (l')$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (m)$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \operatorname{vers}^{-1} \frac{x}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (m')$$

## Examples I.

Find the values of the following integrals :

$$1. \int \frac{dx}{\sqrt{x}}, \quad 2\sqrt{x}.$$

$$2. \int \frac{dx}{x^2}, \quad -\frac{1}{x}.$$

$$3. \int_1 \frac{dx}{x^{\frac{3}{2}}}, \quad 2 - \frac{2}{\sqrt{x}}.$$

$$4. \int_1^{\infty} \frac{dx}{x^5}, \quad \frac{1}{4}.$$

$$5. \int_0 \sqrt{x} dx, \quad \frac{2}{3}x^{\frac{3}{2}}.$$

$$6. \int_1 (x-1)^2 dx, \quad \frac{x^3}{3} - x^2 + x - \frac{1}{3}.$$

$$7. \int_0^{\frac{a}{b}} (a-bx)^2 dx, \quad a^2x - abx + \frac{b^2x^3}{3} \Big|_0^{\frac{a}{b}} = \frac{a^3}{3b}.$$

$$8. \int_{-a}^a (a+x)^3 dx, \quad a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4} \Big|_{-a}^a = 4a^4.$$

$$9. \int_1^{a^2} \frac{dx}{x}, \quad 2 \log a.$$

$$10. \int_{-1}^{-2} \frac{dx}{x}, \quad \log(-x) \Big|_{-1}^{-2} = \log 2.$$

$$11. \int_a^{4a} \frac{(a+x)^2}{\sqrt{x}} dx, \quad 2 \sqrt{x} (a^2 + \frac{2}{3}ax + \frac{1}{5}x^2) \Big|_a^{4a} = 23\frac{1}{5} \cdot a^{\frac{5}{2}}.$$

$$12. \int_0^y \varepsilon^x dx, \quad \varepsilon^y - 1.$$

$$13. \int_0^\theta \sin \theta d\theta, \quad 1 - \cos \theta.$$

$$14. \int_0^\pi \cos x dx, \quad \sin x \Big|_0^\pi = 0.$$

$$15. \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\cos^2 \theta}, \quad \tan \theta \Big|_0^{\frac{1}{2}\pi} = 1.$$

$$16. \int_0^{1a} \frac{dx}{\sqrt{a^2 - x^2}}, \quad \sin^{-1} \frac{x}{a} \Big|_0^{1a} = \frac{\pi}{6}.$$

$$17. \int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}, \quad \frac{\pi}{a}.$$

$$18. \int_1^\infty \frac{dx}{x \sqrt{x^2 - 1}}, \quad \frac{\pi}{2}.$$

19. If a body is projected vertically upward, its velocity after  $t$  units of time is expressed by

$$v = a - gt,$$

$a$  denoting the initial velocity; find the space  $s_1$  described in the time  $t_1$  and the greatest height to which the body will rise.

$$s_1 = \int_0^{t_1} v dt = at_1 - \frac{1}{2}gt_1^2,$$

$$\text{when } v = 0, t = \frac{a}{g}, s = \frac{a^2}{2g}.$$

20. If the velocity of a pendulum is expressed by

$$v = a \cos \frac{\pi t}{2\tau},$$

the position corresponding to  $t = 0$  being taken as origin, find an expression for its position  $s$  at the time  $t$ , and the extreme positive and negative values of  $s$ .

$$s = \frac{2\tau a}{\pi} \sin \frac{\pi t}{2\tau}.$$

$$s = \pm \frac{2\tau a}{\pi} \text{ when } t = \tau, 3\tau, 5\tau, \text{ etc.}$$

21. Find the area included between the axis of  $x$  and a branch of the curve

$$y = \sin x. \quad 2.$$

22. Show that the area between the axis of  $x$ , the parabola

$$y^2 = 4ax,$$

and any ordinate is two thirds of the rectangle whose sides are the ordinate and the corresponding abscissa.

23. Find  $(\alpha)$  the area included by the axes, the curve

$$y = e^x,$$

and the ordinate corresponding to  $x = 1$ , and  $(\beta)$  the whole area between the curve and axes on the left of the axis of  $y$ .

$$(\alpha) e - 1, (\beta) 1.$$

24. Find the area between the parabola of the  $n$ th degree,

$$a^{n-1}y = x^n,$$

and the coordinates of the point  $(a, a)$ .

$$\frac{a^2}{n+1}.$$



25. Show that the area between the axis of  $x$ , the rectangular hyperbola

$$xy = 1,$$

the ordinate corresponding to  $x = 1$ , and any other ordinate is equivalent to the Napierian logarithm of the abscissa of the latter ordinate.

*For this reason Napierian logarithms are often called hyperbolic logarithms.*

26. Find the whole area between the axes, the curve

$$y^n x^m = a^{n+m},$$

and the ordinate for  $x = a$ ,  $m$  and  $n$  being positive.

$$\text{If } n > m, \quad \frac{na^2}{n-m};$$

$$\text{if } n \leq m, \quad \infty.$$

27. If the ordinate  $BR$  of any point  $B$  on the circle

$$x^2 + y^2 = a^2$$

be produced so that  $BR \cdot RP = a^2$ , prove that the whole area between the locus of  $P$  and its asymptotes is double the area of the circle.

28. Find the whole area between the axis of  $x$  and the curve

$$y(a^2 + x^2) = a^3. \quad \pi a^2.$$

29. Find the area between the axis of  $x$  and one branch of the *companion to the cycloid*, the equations of which are

$$x = a\psi \quad y = a(1 - \cos \psi). \quad 2\pi a^2.$$

## II.

*Direct Integration.*

II. In any one of the formulas of Art. 10, we may of course substitute for  $x$  and  $dx$  any function of  $x$  and its differential. For instance, if in formula (b) we put  $x - a$  in place of  $x$ , we have

$$\int \frac{dx}{x-a} = \log (x-a) \quad \text{or} \quad \log (a-x),$$

according as  $x$  is greater or less than  $a$ .

When a given integral is obviously the result of such a substitution in one of the fundamental integrals, or can be made to take this form by the introduction of a constant factor, it is said to be *directly integrable*. Thus,  $\int \sin mx \, dx$  is directly integrable by formula (e); for, if in this formula we put  $mx$  for  $\theta$ , we have

$$\int \sin mx \cdot m \, dx = -\cos mx,$$

hence

$$\int \sin mx \, dx = \frac{1}{m} \int \sin mx \cdot m \, dx = -\frac{1}{m} \cos mx.$$

So also in 
$$\int \sqrt{a + bx^2} \, x \, dx,$$

the quantity  $x \, dx$  becomes the differential of the binomial  $(a + bx^2)$  when we introduce the constant factor  $2b$ , hence this integral can be converted into the result obtained by putting  $(a + bx^2)$  in place of  $x$  in  $\int \sqrt{x} \, dx$ , which is a case of formula (a).

Thus

$$\int \sqrt{a + bx^2} \, x \, dx = \frac{1}{2b} \int (a + bx^2)^{\frac{1}{2}} 2bx \, dx = \frac{1}{3b} (a + bx^2)^{\frac{3}{2}}.$$

12. A simple algebraic or trigonometric transformation sometimes suffices to render an expression directly integrable, or to separate it into directly integrable parts. Thus, since  $-\sin x \, dx$  is the differential of  $\cos x$ , we have by formula (b)

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\log \cos x.$$

So also, by formula (f),

$$\int \tan^2 \theta \, d\theta = \int (\sec^2 \theta - 1) \, d\theta = \tan \theta - \theta;$$

by (e) and (a),

$$\int \sin^3 \theta \, d\theta = \int (1 - \cos^2 \theta) \sin \theta \, d\theta = -\cos \theta + \frac{1}{3} \cos^3 \theta;$$

by (j) and (a),

$$\begin{aligned} \int \sqrt{\left(\frac{1+x}{1-x}\right)} dx &= \int \frac{1+x}{\sqrt{1-x^2}} dx \\ &= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x \, dx) = \sin^{-1} x - \sqrt{1-x^2}. \end{aligned}$$

### *Rational Fractions.*

13. When the coefficient of  $dx$  in an integral is a fraction whose terms are rational functions of  $x$ , the integral may generally be separated into parts directly integrable. If the denominator is of the first degree, we proceed as in the following example.

Given the integral  $\int \frac{x^2 - x + 3}{2x + 1} dx;$

by division,

$$\frac{x^2 - x + 3}{2x + 1} = \frac{x}{2} - \frac{3}{4} + \frac{15}{4} \frac{1}{2x + 1},$$

hence

$$\begin{aligned}\int \frac{x^2 - x + 3}{2x + 1} dx &= \frac{1}{2} \int x dx - \frac{3}{4} \int dx + \frac{15}{4} \int \frac{dx}{2x + 1} \\ &= \frac{x^2}{4} - \frac{3x}{4} + \frac{15}{8} \log (2x + 1).\end{aligned}$$

When the denominator is of higher degree, it is evident that we may, by division, make the integration depend upon that of a fraction in which the degree of the numerator is lower than that of the denominator by at least a unit. We shall consider therefore fractions of this form only.

### *Denominators of the Second Degree.*

**14.** If the denominator is of the second degree, it will (after removing a constant, if necessary) either be the square of an expression of the first degree, or else such a square increased or diminished by a constant. As an example of the first case, let us take

$$\int \frac{x + 1}{(x - 1)^2} dx.$$

The fraction may be decomposed thus :

$$\frac{x + 1}{(x - 1)^2} = \frac{x - 1 + 2}{(x - 1)^2} = \frac{1}{x - 1} + \frac{2}{(x - 1)^2};$$

hence

$$\begin{aligned}\int \frac{x + 1}{(x - 1)^2} dx &= \int \frac{dx}{x - 1} + 2 \int \frac{dx}{(x - 1)^2} \\ &= \log (x - 1) - \frac{2}{x - 1}.\end{aligned}$$

**15.** The integral 
$$\int \frac{x + 3}{x^2 + 2x + 6} dx$$

affords an example of the second case; for the denominator may be written in the form

$$x^2 + 2x + 6 = (x + 1)^2 + 5.$$

Decomposing the fraction as in the preceding article,

$$\frac{x + 3}{(x + 1)^2 + 5} = \frac{x + 1}{(x + 1)^2 + 5} + \frac{2}{(x + 1)^2 + 5};$$

whence

$$\int \frac{x + 3}{x^2 + 2x + 6} dx = \int \frac{(x + 1) dx}{(x + 1)^2 + 5} + 2 \int \frac{dx}{(x + 1)^2 + 5}.$$

The first of the integrals in the second member is directly integrable by formula (b), since the differential of the denominator is  $2(x + 1) dx$ , and the second is a case of formula (k'). Therefore

$$\int \frac{x + 3}{x^2 + 2x + 6} dx = \frac{1}{2} \log (x^2 + 2x + 6) + \frac{2}{\sqrt{5}} \tan^{-1} \frac{x + 1}{\sqrt{5}}.$$

16. To illustrate the third case, let us take

$$\int \frac{2x + 1}{x^2 - x - 6} dx,$$

in which the denominator is equivalent to  $(x - \frac{1}{2})^2 - 6\frac{1}{4}$ , and can therefore be resolved into real factors of the first degree. We can then decompose the fraction into fractions having these factors for denominators. Thus, in the present example, assume

$$\frac{2x + 1}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}, \quad \dots \dots (1)$$

in which  $A$  and  $B$  are numerical quantities to be determined. Multiplying by  $(x - 3)(x + 2)$ ,

$$2x + 1 = A(x + 2) + B(x - 3). \quad \dots \dots (2)$$

Since equation (2) is an algebraic identity, we may in it assign any value we choose to  $x$ . Putting  $x = 3$ , we find

$$7 = 5A, \quad \text{whence} \quad A = \frac{7}{5},$$

putting  $x = -2$ ,

$$-3 = -5B, \quad \text{whence} \quad B = \frac{3}{5}.$$

Substituting these values in (1),

$$\frac{2x+1}{x^2-x-6} = \frac{7}{5(x-3)} + \frac{3}{5(x+2)},$$

whence

$$\int \frac{2x+1}{x^2-x-6} dx = \frac{7}{5} \log(x-3) + \frac{3}{5} \log(x+2).$$

**17.** If the denominator, in a case of the kind last considered, is denoted by  $(x-a)(x-b)$ ,  $a$  and  $b$  are evidently the roots of the equation formed by putting this denominator equal to zero. The cases considered in Art. 14 and Art. 15 are respectively those in which the roots of this equation are equal, and those in which the roots are imaginary. When the roots are real and unequal, if the numerator does not contain  $x$ , the integral can be reduced to the form

$$\int \frac{dx}{(x-a)(x-b)},$$

and by the method given in the preceding article we find

$$\begin{aligned} \int \frac{dx}{(x-a)(x-b)} &= \frac{1}{a-b} \left[ \log(x-a) - \log(x-b) \right] \\ &= \frac{1}{a-b} \log \frac{x-a}{x-b}, \quad . \quad . \quad . \quad . \quad . \quad (A)^* \end{aligned}$$

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\* The formulæ of this series are collected together at the end of Chapter II. for convenience of reference. See Art. 101.

in which, when  $x < a$ ,  $\log(a - x)$  should be written in place of  $\log(x - a)$ . [See note on formula (b), Art. 10.]

If  $b = -a$ , this formula becomes

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} \quad . \quad . \quad . \quad . \quad (A')$$

Integrals of the special forms given in (A) and (A') may be evaluated by the direct application of these formulas. Thus, given the integral

$$\int \frac{dx}{2x^2 + 3x - 2};$$

if we place the denominator equal to zero, we have the roots  $a = \frac{1}{2}$ ,  $b = -2$ ; whence by formula (A),

$$\int \frac{dx}{2x^2 + 3x - 2} = \frac{1}{2} \int \frac{dx}{(x - \frac{1}{2})(x + 2)} = \frac{1}{2} \cdot \frac{1}{2\frac{1}{2}} \log \frac{x - \frac{1}{2}}{x + 2};$$

or, since  $\log(2x - 1)$  differs from  $\log(x - \frac{1}{2})$  only by a constant, we may write

$$\int \frac{dx}{2x^2 + 3x - 2} = \frac{1}{5} \log \frac{2x - 1}{x + 2}.$$

### *Denominators of Higher Degree.*

18. When the denominator is of a degree higher than the second, we may in like manner suppose it resolved into factors corresponding to the roots of the equation formed by placing it equal to zero. The fraction (of which we suppose the numerator to be lower in degree than the denominator) may now be decomposed into partial fractions. If the roots are all real and unequal, we assume these partial fractions as in Art. 16; there being one assumed fraction for each factor.

If, however, a pair of imaginary roots occurs, the factor cor-

responding to the pair is of the form  $(x - \alpha)^2 + \beta^2$ , and the partial fraction must be assumed in the form

$$\frac{Ax + B}{(x - \alpha)^2 + \beta^2};$$

for we are only entitled to assume that the numerator of each partial fraction is lower in degree than its denominator (otherwise the given fraction which is the sum of the partial fractions would not have this property). For example, given

$$\int \frac{x + 3}{(x^2 + 1)(x - 1)} dx.$$

Assume

$$\frac{x + 3}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}, \quad \dots \quad (1)$$

whence

$$x + 3 = (x - 1)(Ax + B) + (x^2 + 1)C.$$

Since in an identical equation the coefficients of the several powers of  $x$  must separately vanish, the coefficients of  $x^2$ ,  $x$  and  $x^0$  give, for the determination of  $A$ ,  $B$  and  $C$ , the three equations

$$A + C = 0, \quad B - A = 1, \quad C - B = 3.$$

From these we obtain  $A = -2$ ,  $B = -1$ ,  $C = 2$ ; hence, substituting in equation (1),

$$\frac{x + 3}{(x^2 + 1)(x - 1)} = \frac{2}{x - 1} - \frac{2x + 1}{x^2 + 1},$$

therefore

$$\begin{aligned} \int \frac{x + 3}{(x^2 + 1)(x - 1)} dx &= 2 \int \frac{dx}{x - 1} - \int \frac{2x dx}{x^2 + 1} - \int \frac{dx}{x^2 + 1} \\ &= 2 \log(x - 1) - \log(x^2 + 1) - \tan^{-1} x. \end{aligned}$$

**19.** The method of determining the assumed coefficients illustrated above makes it evident that, the denominator being



of the  $n$ th degree, we must assume  $n$  of these coefficients; because we have to satisfy  $n$  equations, derived from the powers of  $x$  from  $x^{n-1}$  to  $x^0$ .

It is evident that we may take for the denominators of the partial fractions any two or more factors of the given denominator which have no common factor between themselves, provided we assume for each numerator a polynomial of degree just inferior to that of the denominator. But, since our present object is to separate the given fraction into directly integrable parts, when a squared factor such as  $(x - a)^2$  occurs, instead of assuming the corresponding partial fraction in the form  $\frac{Ax + B}{(x - a)^2}$  (which would require to be further decomposed as in Art. 14), we at once assume a pair of fractions of the form

$$\frac{A}{x - a} + \frac{B}{(x - a)^2}.$$

**20.** We proceed in like manner when a higher power of a linear factor occurs. For example, given

$$\int \frac{x + 2}{(x - 1)^3(x + 1)} dx;$$

we assume

$$\frac{x + 2}{(x - 1)^3(x + 1)} = \frac{A}{(x - 1)^3} + \frac{B}{(x - 1)^2} + \frac{C}{x - 1} + \frac{D}{x + 1};$$

whence

$$x + 2 = [A + B(x - 1) + C(x - 1)^2](x + 1) + D(x - 1)^3. \quad (1)$$

Putting  $x = 1$ , we have  $3 = 2A \quad \therefore \quad A = \frac{3}{2}.$

Putting  $x = -1$ , we have  $1 = -8D \quad \therefore \quad D = -\frac{1}{8}.$

The most convenient way to determine the other coefficients is to equate to zero the coefficient of  $x^3$ , and to put  $x = 0$ . We thus obtain

$$0 = C + D, \quad \text{and} \quad 2 = A - B + C - D,$$

from which  $C = \frac{1}{8}$ , and  $B = -\frac{1}{4}$ . Therefore

$$\begin{aligned} \int \frac{x+2}{(x-1)^3(x+1)} dx &= \frac{3}{2} \int \frac{dx}{(x-1)^3} - \frac{1}{4} \int \frac{dx}{(x-1)^2} + \frac{1}{8} \int \frac{dx}{x-1} - \frac{1}{8} \int \frac{dx}{x+1} \\ &= -\frac{3}{4(x-1)^2} + \frac{1}{4(x-1)} + \frac{1}{8} \log \frac{x-1}{x+1}. \end{aligned}$$

**21.** The fraction corresponding to a simple factor of the given denominator may be found, independently of the other partial fractions, by means of the expression derived below. Denote the given denominator by  $\phi(x)$ , and let it contain the simple factor  $x - a$ , so that

$$\phi(x) = (x - a)\psi(x), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which  $\psi(x)$  does not vanish when  $x = a$ .

Let  $f(x)$  denote the numerator (it is not necessary here to suppose this to be lower than  $\phi(x)$  in degree), and let  $Q$  denote the entire part of the quotient. Then we may assume

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x-a)\psi(x)} = Q + \frac{A}{x-a} + \frac{P}{\psi(x)},$$

in which  $Q$  and  $P$  are in general polynomials. Clearing of fractions,

$$f(x) = Q(x-a)\psi(x) + A\psi(x) + P(x-a).$$

Putting  $x = a$ , we have (since neither  $P$  nor  $Q$  can become infinite)

$$f(a) = A\psi(a), \quad \text{whence} \quad A = \frac{f(a)}{\psi(a)}, \quad . \quad . \quad . \quad (2)$$

an expression for the numerator  $A$ .

Again, differentiating equation (1), we have

$$\phi'(x) = \psi(x) + (x-a)\psi'(x);$$



$$x^8 - 1 = (x - 1)(x + 1)(x^2 + 1)(x^2 - x\sqrt{2} + 1)(x^2 + x\sqrt{2} + 1).$$

It is to be noticed that when the fraction is a rational function of some higher power of  $x$ , we may simplify the process of decomposition. Thus it is legitimate to assume

$$\frac{x^4}{x^8 - 1} = \frac{A}{x^4 - 1} + \frac{B}{x^4 + 1},$$

because, if  $z = x^4$ , the numerator and factors of the denominator are rational functions of  $z$ . The first term could be treated in like manner with respect to  $x^2$ ; but not the second, since the factors of  $x^4 + 1$  involve  $x$  as well as  $x^2$ ,

**23.** We have seen in the preceding articles that the partial fractions corresponding to the real roots, whether single or multiple, are directly integrable, and also those corresponding to unrepeatd imaginary roots. In the following example a case of multiple imaginary roots occurs: given

$$\int \frac{a + x}{(a - x)(a^2 + x^2)^2} dx.$$

It is readily shown that the partial fraction corresponding to  $a - x$  is  $\frac{1}{2a^3(a - x)}$ , and the remainder is conveniently found by subtracting this from the given fraction; thus

$$\frac{a + x}{(a - x)(a^2 + x^2)^2} - \frac{1}{2a^3(a - x)} = \frac{a^4 + 2a^3x - 2a^2x^2 - x^4}{2a^3(a - x)(a^2 + x^2)^2}.$$

The numerator of this remainder is found to be divisible by  $a - x$ , thus verifying the process; and the given integral reduces to

$$- \frac{1}{2a^3} \log(a - x) + \frac{1}{2a^3} \int \frac{a^3 + 3a^2x + ax^2 + x^3}{(a^2 + x^2)^2} dx.$$

The integration of the last term may be effected by a transformation given later. See Art. 41.

**24.** Instead of assuming the partial fractions with undeter-

mined numerators, it is sometimes possible to proceed more expeditiously as in the following examples:

Given

$$\int \frac{1}{x^3(1+x^2)} dx;$$

putting the numerator in the form  $1 + x^2 - x^2$ , we have

$$\begin{aligned} \int \frac{1}{x^3(1+x^2)} dx &= \int \frac{1+x^2}{x^3(1+x^2)} dx - \int \frac{x^2}{x^3(1+x^2)} dx \\ &= \int \frac{dx}{x^3} - \int \frac{1}{x(1+x^2)} dx. \end{aligned}$$

Treating the last integral in like manner,

$$\begin{aligned} \int \frac{1}{x^3(1+x^2)} dx &= \int \frac{dx}{x^3} - \int \frac{dx}{x} + \int \frac{x dx}{1+x^2} \\ &= -\frac{1}{2x^2} - \log x + \frac{1}{2} \log(1+x^2) = -\frac{1}{2x^2} + \log \frac{\sqrt{1+x^2}}{x}. \end{aligned}$$

Again, given

$$\int \frac{1}{x^2(1+x)^2} dx;$$

putting the numerator in the form  $(1+x)^2 - 2x - x^2$ , we have

$$\begin{aligned} \int \frac{1}{x^2(1+x)^2} dx &= \int \frac{dx}{x^2} - \int \frac{2+x}{x(1+x)^2} dx \\ &= \int \frac{dx}{x^2} - 2 \int \frac{dx}{x(1+x)} + \int \frac{dx}{(1+x)^2}. \end{aligned}$$

Hence by equation (A), Art. 17,

$$\int \frac{dx}{x^2(1+x)^2} = -\frac{1}{x} - 2 \log \frac{x}{1+x} - \frac{1}{1+x}.$$

## Examples II.

1.  $\int \frac{dx}{a-x},$   $-\log(a-x).$
2.  $\int \frac{dx}{(a-x)^2},$   $\frac{1}{a-x},$
3.  $\int \frac{x dx}{a^2+x^2},$   $\frac{1}{2} \log(a^2+x^2).$
4.  $\int_0^1 \sqrt{(a^2-x^2)} x dx,$   $\frac{a^3 - (a^2-x^2)^{\frac{3}{2}}}{3},$
5.  $\int_0^1 \frac{x^2 dx}{a^3-x^3},$   $\frac{1}{3} \log \frac{a^3}{a^3-x^3}.$
6.  $\int_0^1 \frac{x dx}{\sqrt{(a^2-x^2)}},$   $a - \sqrt{(a^2-x^2)}.$
7.  $\int (a^2+3x^2)^3 x dx,$   $\frac{(a^2+3x^2)^4}{24}.$
8.  $\int_0^1 (a+mx)^2 dx,$   $\frac{(a+mx)^3 - a^3}{3m}.$
9.  $\int \frac{dx}{\sin^2 2x},$   $-\frac{\cot 2x}{2}.$
10.  $\int_0^1 \cos^3 x \sin x dx,$   $\frac{1 - \cos^4 x}{4}.$
11.  $\int \frac{\cos \theta d\theta}{\sin^3 \theta},$   $-\frac{1}{2} \operatorname{cosec}^2 \theta.$
12.  $\int_0^1 \sec^3 3x \tan 3x dx,$   $\frac{\sec^3 3x - 1}{9}.$

$$13. \int a^{mx} dx, \quad \frac{a^{mx}}{m \log a}.$$

$$14. \int (\varepsilon^x - 1)^3 dx, \quad \frac{1}{3} \varepsilon^{3x} - \frac{3}{2} \varepsilon^{2x} + 3 \varepsilon^x - x.$$

$$15. \int (1 + 3 \sin^2 x)^2 \sin x \cos x dx, \quad \frac{(1 + 3 \sin^2 x)^3}{18}.$$

$$16. \int_0^{2a} \frac{(a-x) dx}{\sqrt{(2ax-x^2)}}, \quad \sqrt{(2ax-x^2)} \Big|_0^{2a} = 0.$$

$$17. \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta, \quad \frac{2}{3}.$$

$$18. \int \sec^4 \theta d\theta, \quad \tan \theta + \frac{1}{3} \tan^3 \theta.$$

$$19. \int \tan^3 x dx, \quad \frac{1}{2} \tan^2 x + \log \cos x.$$

$$20. \int_0^{\frac{\pi}{4}} \sec^4 x \tan x dx, \quad \frac{1}{4} \sec^4 x \Big|_0^{\frac{\pi}{4}} = \frac{3}{4}.$$

$$21. \int \sqrt{\frac{a-x}{a+x}} dx, \quad a \sin^{-1} \frac{x}{a} + \sqrt{(a^2 - x^2)}.$$

$$22. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 \theta d\theta, \quad \frac{1 - \log 2}{2}.$$

$$23. \int \sqrt{\frac{2a-x}{x}} dx, \quad \sqrt{(2ax-x^2)} + a \operatorname{vers}^{-1} \frac{x}{a}.$$

$$24. \int \sin(\alpha - 2\theta) d\theta, \quad \frac{\cos(\alpha - 2\theta)}{2}.$$

$$25. \int \frac{\cos x \, dx}{a - b \sin x}, \quad -\frac{1}{b} \log(a - b \sin x).$$

$$26. \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\tan x}, \quad \frac{1}{2} \log 2.$$

$$27. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{\tan x}, \quad \frac{1}{2} \log 2.$$

$$28. \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{dx}{x \log x}, \quad \log(-\log x) \Big|_{\frac{1}{4}}^{\frac{1}{2}} = -\log 2.$$

$$29. \int \frac{dx}{e^x + e^{-x}}, \quad \tan^{-1} e^x.$$

$$30. \int \frac{x^2 \, dx}{x^6 + 1}, \quad \frac{1}{3} \tan^{-1} x^3.$$

$$31. \int \frac{x \, dx}{\sqrt{a^4 - x^4}}, \quad \frac{1}{2} \sin^{-1} \frac{x^2}{a^2}.$$

$$32. \int \frac{dx}{\sqrt{5 - 3x^2}}, \quad \frac{1}{\sqrt{3}} \sin^{-1} \frac{x \sqrt{3}}{\sqrt{5}}.$$

$$33. \int \frac{dx}{2 + 5x^2}, \quad \frac{1}{\sqrt{10}} \tan^{-1} \frac{x \sqrt{5}}{\sqrt{2}}.$$

$$34. \int_1^{\infty} \frac{dx}{x \sqrt{2x^2 - 1}}, \quad \frac{\pi}{4}.$$

$$35. \int_0^1 \frac{dx}{x^2 + x + 1}, \quad \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} \Big|_0^1 = \frac{\pi}{3\sqrt{3}}.$$



$$36. \int_0^1 \frac{dx}{\sqrt{5-4x-x^2}}, \quad \cos^{-1} \frac{2}{3}.$$

$$37. \int \frac{\sqrt{x^2-a^2}}{x} dx \left[ = \int \frac{x^2-a^2}{x\sqrt{x^2-a^2}} dx \right],$$

$$\sqrt{x^2-a^2} - a \sec^{-1} \frac{x}{a}.$$

$$38. \int_0^{\frac{1}{2}a} \frac{x^2}{a-x} dx, \quad a^2 (\log 2 - \frac{5}{8}).$$

$$39. \int \frac{4x^3-x+3}{x^2+1} dx, \quad 4x - \frac{1}{2} \log(x^2+1) - \tan^{-1} x.$$

$$40. \int \frac{x^2+x+1}{x^2-x+1} dx, \quad x + \log(x^2-x+1) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}.$$

$$41. \int \frac{x^2-1}{x^2-4} dx, \quad x + \frac{3}{4} \log \frac{x-2}{x+2}.$$

$$42. \int \frac{(1+x)^2}{x-x^2} dx, \quad \log \frac{x}{(1-x)^4} - x.$$

$$43. \int \frac{(2x+1)^2 dx}{2x+3}, \quad x^2 - x + 2 \log(2x+3).$$

$$44. \int \frac{2x+3}{(2x+1)^2} dx, \quad \frac{1}{2} \log(2x+1) - \frac{1}{2x+1}.$$

$$45. \int \frac{x^2-3x+3}{x^2-3x+2} dx, \quad x + \log \frac{x-2}{x-1}.$$

$$46. \int_0^a \frac{dx}{x^2-2ax \cos \alpha + a^2},$$

$$\left[ \frac{1}{a \sin \alpha} \tan^{-1} \frac{x-a \cos \alpha}{a \sin \alpha} \right]_0^a = \frac{\pi - \alpha}{2a \sin \alpha}.$$

$$47. \int \frac{dx}{x^2 - 2ax \sec \alpha + a^2}, \quad \frac{1}{2a \tan \alpha} \log \frac{x - a \sec \alpha - a \tan \alpha}{x - a \sec \alpha + a \tan \alpha}.$$

$$48. \int \frac{dx}{2x^2 - 4x - 7}, \quad \frac{\sqrt{2}}{12} \log \frac{2x - 2 - 3\sqrt{2}}{2x - 2 + 3\sqrt{2}}.$$

$$49. \int \frac{x^2 dx}{1 - x^6}, \quad \frac{1}{6} \log \frac{1 + x^3}{1 - x^3}.$$

$$50. \int \frac{3x - 1}{x^3 - x^2 - 2x} dx, \quad \frac{1}{6} \log \frac{x^3(x - 2)}{(x + 1)^6}.$$

$$51. \int \frac{x dx}{(x + 2)(x + 3)^2}, \quad 2 \log \frac{x + 3}{x + 2} - \frac{3}{x + 3}.$$

$$52. \int \frac{x dx}{x^3 + x^2 + x + 1}, \quad \frac{1}{2} \left[ \tan^{-1} x + \log \frac{\sqrt{x^2 + 1}}{x + 1} \right].$$

$$53. \int \frac{x^2 dx}{x^4 + x^2 - 2}, \quad \frac{1}{6} \log \frac{x - 1}{x + 1} + \frac{\sqrt{2}}{3} \tan^{-1} \frac{x}{\sqrt{2}}.$$

$$54. \int \frac{x^2 - x + 2}{x^4 - 5x^2 + 4} dx, \quad \frac{2}{3} \log \frac{x + 1}{x + 2} + \frac{1}{3} \log \frac{x - 2}{x - 1}.$$

$$55. \int \frac{dx}{x^3 - x^2 - x + 1}, \quad \frac{1}{4} \log \frac{x + 1}{x - 1} - \frac{1}{2(x - 1)}.$$

$$56. \int \frac{3x + 1}{x^4 - 1} dx, \quad \log \frac{(x - 1)\sqrt{x + 1}}{(x^2 + 1)^{\frac{3}{4}}} - \frac{1}{2} \tan^{-1} x.$$

$$57. \int \frac{dx}{1 + x^3}, \quad \frac{1}{6} \log \frac{(x + 1)^2}{x^2 - x + 1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}}.$$

$$58. \int \frac{x^2 dx}{(x - 1)^2(x^2 + 1)}, \quad \frac{1}{2} \log(x - 1) - \frac{1}{4} \log(x^2 + 1) - \frac{1}{2(x - 1)}.$$

$$59. \int \frac{dx}{x(1+x+x^2+x^3)},$$

$$\log x - \frac{1}{2} \log(1+x) - \frac{1}{4} \log(1+x^2) - \frac{1}{2} \tan^{-1} x.$$

$$60. \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx,$$

$$\frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1}.$$

$$61. \int \frac{x^2 + x - 1}{x^3 + x^2 - 6x} dx, \quad \frac{1}{6} \log x + \frac{1}{2} \log(x-2) + \frac{1}{3} \log(x+3).$$

$$62. \int \frac{x^2 dx}{x^4 - x^2 - 12},$$

$$\frac{1}{7} \log \frac{x-2}{x+2} + \frac{\sqrt{3}}{7} \tan^{-1} \frac{x}{\sqrt{3}}.$$

$$63. \int \frac{x^2 dx}{(x^2 - 1)^2},$$

$$\frac{1}{4} \log \frac{x-1}{x+1} - \frac{x}{2(x^2 - 1)}.$$

$$64. \int \frac{2x^2 - 3a^2}{x^4 - a^4} dx,$$

$$\frac{5}{2a} \tan^{-1} \frac{x}{a} - \frac{1}{4a} \log \frac{x-a}{x+a}.$$

$$65. \int \frac{x dx}{x^4 - x^2 - 2},$$

$$\frac{1}{6} \log \frac{x^2 - 2}{x^2 + 1}.$$

$$66. \int \frac{dx}{(x^2 + a^2)(x+b)}, \quad \frac{1}{b^2 + a^2} \left[ \log \frac{x+b}{\sqrt{x^2 + a^2}} + \frac{b}{a} \tan^{-1} \frac{x}{a} \right].$$

$$67. \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)},$$

$$\frac{\pi}{2ab(a+b)}.$$

$$68. \int_a^\infty \frac{a^2 dx}{x^2(a^2 + x^2)},$$

$$\frac{4 - \pi}{4a}.$$

$$69. \int \frac{x+1}{x(1+x^2)} dx,$$

$$\tan^{-1} x + \log \frac{x}{\sqrt{1+x^2}}.$$

$$70. \int \frac{dx}{x^4(x^2+1)},$$

$$\tan^{-1} x + \frac{1}{x} - \frac{1}{3x^3}.$$

$$71. \int \frac{dx}{x(1+x)^2}, \quad \log \frac{x}{1+x} + \frac{1}{1+x}.$$

$$72. \int \frac{dx}{x(a+bx^3)}, \quad \frac{1}{3a} \log \frac{x^3}{a+bx^3}.$$

$$73. \int \frac{dx}{x^4(a+bx^3)}, \quad -\frac{1}{3ax^3} + \frac{b}{3a^2} \log \frac{a+bx^3}{x^3}.$$

74. Find the whole area enclosed by both loops of the curve

$$y^2 = x^2(1-x^2). \quad \frac{4}{3}.$$

75. Find the area enclosed between the asymptote corresponding to  $x = a$ , and the curve

$$x^2y^2 + a^2x^2 = a^2y^2. \quad 2a^2.$$

76. Find the whole area enclosed by the curve

$$a^2y^4 = x^4(a^2 - x^2). \quad \frac{8}{5}a^2$$

77. Find the area enclosed by the catenary

$$y = \frac{c}{2} \left[ \epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}} \right],$$

the axes and any ordinate.

$$\frac{c^2}{2} \left[ \epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} \right].$$

78. Find the whole area between the witch

$$xy^2 = 4a^2(2a - x)$$

and its asymptote. See Ex. 23.

$$4\pi a^2.$$

## III.

*Trigonometric Integrals.*

25. The transformation,  $\tan^2 \theta = \sec^2 \theta - 1$ , suffices to separate all integrals of the form

$$\int \tan^n \theta d\theta, \dots \dots \dots (I)$$

in which  $n$  is an integer, into directly integrable parts. Thus, for example,

$$\begin{aligned} \int \tan^5 \theta d\theta &= \int \tan^3 \theta (\sec^2 \theta - 1) d\theta \\ &= \frac{\tan^4 \theta}{4} - \int \tan^3 \theta d\theta. \end{aligned}$$

Transforming the last integral in like manner, we have

$$\int \tan^5 \theta d\theta = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + \int \tan \theta d\theta;$$

hence (see Art. 12)

$$\int \tan^5 \theta d\theta = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} - \log \cos \theta.$$

When the value of  $n$  in (I) is even, the value of the final integral will be  $\theta$ . When  $n$  is negative, the integral takes the form

$$\int \cot^n \theta d\theta,$$

which may be treated in a similar manner.

**26.** Integrals of the form

$$\int \sec^n \theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are readily evaluated *when n is an even number*, thus

$$\begin{aligned} \int \sec^6 \theta d\theta &= \int (\tan^2 \theta + 1)^2 \sec^2 \theta d\theta \\ &= \int \tan^4 \theta \sec^2 \theta d\theta + 2 \int \tan^2 \theta \sec^2 \theta d\theta + \int \sec^2 \theta d\theta \\ &= \frac{\tan^5 \theta}{5} + \frac{2 \tan^3 \theta}{3} + \tan \theta. \end{aligned}$$

If  $n$  in expression (2) is odd, the method to be explained in Section VI is required.

Integrals of the form  $\int \operatorname{cosec}^n \theta d\theta$  are treated in like manner.

*Cases in which  $\sin^m \theta \cos^n \theta d\theta$  is directly integrable.*

**27.** If  $n$  is a *positive odd number*, an integral of the form

$$\int \sin^m \theta \cos^n \theta d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is directly integrable in terms of  $\sin \theta$ . Thus,

$$\begin{aligned} \int \sin^2 \theta \cos^5 \theta d\theta &= \int \sin^2 \theta (1 - \sin^2 \theta)^2 \cos \theta d\theta \\ &= \frac{\sin^3 \theta}{3} - \frac{2 \sin^5 \theta}{5} + \frac{\sin^7 \theta}{7}. \end{aligned}$$

This method is evidently applicable even when  $m$  is fractional or negative. Thus, putting  $y$  for  $\sin \theta$ ,

$$\int \frac{\cos^3 \theta}{\sin^{\frac{3}{2}} \theta} d\theta = \int \frac{(1 - y^2) dy}{y^{\frac{3}{2}}} = \int y^{-\frac{3}{2}} dy - \int y^{\frac{1}{2}} dy;$$

hence

$$\int \frac{\cos^3 \theta}{\sin^{\frac{3}{2}} \theta} d\theta = -2y^{-\frac{1}{2}} - \frac{2}{3} y^{\frac{3}{2}} = -\frac{2}{3} \cdot \frac{3 + \sin^2 \theta}{\sqrt{\sin \theta}}.$$

When  $m$  in expression (3) is a positive odd number, the integral is evaluated in a similar manner.

**28.** An integral of the form (3) is also directly integrable when  $m + n$  is an even negative integer, in other words, when it can be written in the form

$$\int \frac{\sin^m \theta d\theta}{\cos^{m+2q} \theta} = \int \tan^m \theta \sec^{2q} \theta d\theta,$$

in which  $q$  is positive.

For example,

$$\begin{aligned} \int \frac{d\theta}{\sin^{\frac{3}{2}} \theta \cos^{\frac{5}{2}} \theta} &= \int (\tan \theta)^{-\frac{3}{2}} \sec^4 \theta d\theta \\ &= \int (\tan \theta)^{-\frac{3}{2}} (\tan^2 \theta + 1) \sec^2 \theta d\theta; \end{aligned}$$

hence

$$\int \frac{d\theta}{\sin^{\frac{3}{2}} \theta \cos^{\frac{5}{2}} \theta} = \frac{2}{3} \tan^{\frac{3}{2}} \theta - \frac{2}{\tan^{\frac{1}{2}} \theta}.$$

It may be more convenient to express the integral in terms of  $\cot \theta$  and  $\operatorname{cosec} \theta$ , thus

$$\begin{aligned} \int \frac{\cos^4 \theta d\theta}{\sin^8 \theta} &= \int \cot^4 \theta (\cot^2 \theta + 1) \operatorname{cosec}^2 \theta d\theta \\ &= -\frac{\cot^7 \theta}{7} - \frac{\cot^5 \theta}{5}. \end{aligned}$$

Integrals of the forms treated in Art. 25 and Art. 26 are included in the general form (3), Art. 27. Except in the cases already considered, and in the special cases given below, the method of reduction given in Section VI is required in the evaluation of integrals of this form.

*The Integrals  $\int \sin^2 \theta d\theta$ , and  $\int \cos^2 \theta d\theta$ .*

29. These integrals are readily evaluated by means of the transformations

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \text{and} \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta).$$

Thus

$$\int \sin^2 \theta d\theta = \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta = \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta,$$

or, since  $\sin 2\theta = 2 \sin \theta \cos \theta$ ,

$$\int \sin^2 \theta d\theta = \frac{1}{2}(\theta - \sin \theta \cos \theta). \quad . \quad . \quad . \quad (B)$$

In like manner

$$\int \cos^2 \theta d\theta = \frac{1}{2}(\theta + \sin \theta \cos \theta). \quad . \quad . \quad . \quad (C)$$

Since  $\sin^2 \theta + \cos^2 \theta = 1$ , the sum of these integrals is  $\int d\theta$ ; accordingly we find the sum of their values to be  $\theta$ .

In the applications of the Integral Calculus, these integrals frequently occur with the limits 0 and  $\frac{1}{2}\pi$ ; from (B) and (C) we derive

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{1}{4}\pi.$$



The Integrals  $\int \frac{d\theta}{\sin \theta \cos \theta}$ ,  $\int \frac{d\theta}{\sin \theta}$ , and  $\int \frac{d\theta}{\cos \theta}$ .

30. We have

$$\int \frac{d\theta}{\sin \theta \cos \theta} = \int \frac{\sec^2 \theta d\theta}{\tan \theta} = \log \tan \theta. \quad \dots (D)$$

Again, using the transformation,

$$\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta,$$

we have

$$\int \frac{d\theta}{\sin \theta} = \int \frac{\frac{1}{2} d\theta}{\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} = \int \frac{\sec^2 \frac{1}{2}\theta \frac{1}{2} d\theta}{\tan \frac{1}{2}\theta};$$

hence

$$\int \frac{d\theta}{\sin \theta} = \log \tan \frac{1}{2}\theta. \quad \dots (E)$$

This integral may also be evaluated thus,

$$\int \frac{d\theta}{\sin \theta} = \int \frac{\sin \theta d\theta}{\sin^2 \theta} = \int \frac{\sin \theta d\theta}{1 - \cos^2 \theta}.$$

Since  $\sin \theta d\theta = -d(\cos \theta)$ , the value of the last integral is, by formula (A'), Art. 17,

$$\frac{1}{2} \log \frac{1 - \cos \theta}{1 + \cos \theta} = \log \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}};$$

and, multiplying both terms of the fraction by  $1 - \cos \theta$ , we have

$$\int \frac{d\theta}{\sin \theta} = \log \frac{1 - \cos \theta}{\sin \theta}. \quad \dots (E')$$

31. Since  $\cos \theta = \sin (\frac{1}{2}\pi + \theta)$ , we derive from formula (E),

$$\int \frac{d\theta}{\cos \theta} = \int \frac{d\theta}{\sin (\frac{1}{2}\pi + \theta)} = \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right]. \quad (F)$$

By employing a process similar to that used in deriving formula (E'), we have also

$$\int \frac{d\theta}{\cos \theta} = \log \frac{1 + \sin \theta}{\cos \theta}. \quad (F')$$

### *Miscellaneous Trigonometric Integrals.*

32. A trigonometric integral may sometimes be reduced, by means of the formulas for trigonometric transformation, to one of the forms integrated in the preceding articles. For example, let us take the integral

$$\int \frac{d\theta}{a \sin \theta + b \cos \theta}.$$

Putting  $a = k \cos \alpha$ ,  $b = k \sin \alpha$ , . . . . (I)

we have

$$\int \frac{d\theta}{a \sin \theta + b \cos \theta} = \frac{1}{k} \int \frac{d\theta}{\sin (\theta + \alpha)}.$$

Hence by formula (E)

$$\int \frac{d\theta}{a \sin \theta + b \cos \theta} = \frac{1}{k} \log \tan \frac{1}{2}(\theta + \alpha);$$

or, since equations (I) give

$$k = \sqrt{(a^2 + b^2)}, \quad \tan \alpha = \frac{b}{a},$$

$$\int \frac{d\theta}{a \sin \theta + b \cos \theta} = \frac{1}{\sqrt{(a^2 + b^2)}} \log \tan \frac{1}{2} \left[ \theta + \tan^{-1} \frac{b}{a} \right]$$

33. The expression  $\sin m\theta \sin n\theta d\theta$  may be integrated by means of the formula

$$\cos (m-n) \theta - \cos (m+n) \theta = 2 \sin m\theta \sin n\theta ;$$

whence

$$\int \sin m\theta \sin n\theta d\theta = \frac{\sin (m-n) \theta}{2 (m-n)} - \frac{\sin (m+n) \theta}{2 (m+n)} . \quad (1)$$

In like manner, from

$$\cos (m-n) \theta + \cos (m+n) \theta = 2 \cos m\theta \cos n\theta ,$$

we derive

$$\int \cos m\theta \cos n\theta d\theta = \frac{\sin (m-n) \theta}{2 (m-n)} + \frac{\sin (m+n) \theta}{2 (m+n)} . \quad (2)$$

When  $m = n$ , the first term of the second member of each of these equations takes an indeterminate form. Evaluating this term, we have

$$\int \sin^2 n\theta d\theta = \frac{\theta}{2} - \frac{\sin 2n\theta}{4n} , \quad . \quad . \quad . \quad (3)$$

and 
$$\int \cos^2 n\theta d\theta = \frac{\theta}{2} + \frac{\sin 2n\theta}{4n} . \quad . \quad . \quad . \quad (4)$$

Using the limits 0 and  $\pi$  we have, from (1) and (2), *when m and n are unequal integers*,

$$\int_0^\pi \sin m\theta \sin n\theta d\theta = \int_0^\pi \cos m\theta \cos n\theta d\theta = 0 ; \quad . \quad . \quad (5)$$

but, when  $m$  and  $n$  are *equal integers*, we have from (3) and (4)

$$\int_0^\pi \sin^2 n\theta d\theta = \int_0^\pi \cos^2 n\theta d\theta = \frac{\pi}{2} . \quad . \quad . \quad . \quad (6)$$

34. To integrate  $\sqrt{1 + \cos \theta} d\theta$ , we use the formula

$$2 \cos^2 \frac{1}{2} \theta = 1 + \cos \theta .$$

whence  $\sqrt{1 + \cos \theta} = \pm \sqrt{2} \cos \frac{1}{2}\theta$ ,

in which the positive sign is to be taken, provided the value of  $\theta$  is between 0 and  $\pi$ . Supposing this to be the case, we have

$$\begin{aligned} \int \sqrt{1 + \cos \theta} d\theta &= \sqrt{2} \int \cos \frac{1}{2}\theta d\theta \\ &= 2\sqrt{2} \sin \frac{1}{2}\theta. \end{aligned}$$

For example, we have the definite integral

$$\int_0^{\frac{\pi}{2}} \sqrt{1 + \cos \theta} d\theta = 2\sqrt{2} \sin \frac{\pi}{4} = 2.$$

*Integration of  $\frac{d\theta}{a + b \cos \theta}$ .*

**35.** By means of the formulas

$$1 = \cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta \quad \text{and} \quad \cos \theta = \cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta,$$

we have

$$\int \frac{d\theta}{a + b \cos \theta} = \int \frac{d\theta}{(a + b) \cos^2 \frac{1}{2}\theta + (a - b) \sin^2 \frac{1}{2}\theta}.$$

Multiplying numerator and denominator by  $\sec^2 \frac{1}{2}\theta$ , this becomes

$$\int \frac{\sec^2 \frac{1}{2}\theta d\theta}{a + b + (a - b) \tan^2 \frac{1}{2}\theta},$$

and, putting for abbreviation

$$\tan \frac{1}{2}\theta = y,$$

we have, since  $\frac{1}{2} \sec^2 \frac{1}{2}\theta d\theta = dy$ ,

$$\int \frac{d\theta}{a + b \cos \theta} = 2 \int \frac{dy}{a + b + (a - b)y^2}.$$

The form of this integral depends upon the relative values of  $a$  and  $b$ . Assuming  $a$  to be positive, if  $b$ , which may be either positive or negative, is numerically less than  $a$ , we may put

$$\frac{a \pm b}{a - b} = c^2.$$

The integral may then be written in the form

$$\frac{2}{a - b} \int \frac{dy}{c^2 + y^2},$$

the value of which is, by formula ( $k'$ ),

$$\frac{2}{c(a - b)} \tan^{-1} \frac{y}{c}.$$

Hence, substituting their values for  $y$  and  $c$ , we have, in this case,

$$\int \frac{d\theta}{a + b \cos \theta} = \frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left[ \sqrt{\frac{a - b}{a + b}} \tan \frac{1}{2} \theta \right]. \quad (G)$$

If, on the other hand,  $b$  is numerically greater than  $a$ , this expression for the integral involves imaginary quantities; but putting

$$\frac{b + a}{b - a} = c^2,$$

the integral becomes

$$\frac{2}{b - a} \int \frac{dy}{c^2 - y^2},$$

the value of which is, by formula ( $A'$ ), Art. 17,

$$\frac{1}{c(b - a)} \log \frac{c + y}{c - y}.$$

Therefore, in this case,

$$\int \frac{d\theta}{a + b \cos \theta} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b+a} + \sqrt{b-a} \tan \frac{1}{2} \theta}{\sqrt{b+a} - \sqrt{b-a} \tan \frac{1}{2} \theta}. \quad (G')$$

**36.** If  $e < 1$ , formula (G) of the preceding article gives

$$\int \frac{d\theta}{1 + e \cos \theta} = \frac{2}{\sqrt{1 - e^2}} \tan^{-1} \left[ \sqrt{\frac{1 - e}{1 + e}} \tan \frac{1}{2} \theta \right]. \quad (I)$$

Putting

$$\sqrt{\frac{1 - e}{1 + e}} \cdot \tan \frac{1}{2} \theta = \tan \frac{1}{2} \phi, \quad \dots \dots \dots (2)$$

and noticing that  $\phi = 0$  when  $\theta = 0$ , we may write

$$\int_0 \frac{d\theta}{1 + e \cos \theta} = \frac{\phi}{\sqrt{1 - e^2}}. \quad \dots \dots \dots (3)$$

Now, if in equation (I) we put  $\phi$  for  $\theta$  and change the sign of  $e$ , we obtain

$$\int_0 \frac{d\phi}{1 - e \cos \phi} = \frac{2}{\sqrt{1 - e^2}} \tan^{-1} \left[ \sqrt{\frac{1 + e}{1 - e}} \tan \frac{1}{2} \phi \right];$$

hence, by equation (2),

$$\int_0 \frac{d\phi}{1 - e \cos \phi} = \frac{\theta}{\sqrt{1 - e^2}}. \quad \dots \dots \dots (4)$$

Equations (3) and (4) are equivalent to

$$\frac{d\theta}{1 + e \cos \theta} = \frac{d\phi}{\sqrt{1 - e^2}}, \quad \dots \dots \dots (5)$$

and

$$\frac{d\phi}{1 - e \cos \phi} = \frac{d\theta}{\sqrt{1 - e^2}}, \quad \dots \dots \dots (6)$$

the product of which gives

$$(1 + e \cos \theta) (1 - e \cos \phi) = 1 - e^2 \quad (7)$$

By means of these relations any expression of the form

$$\int \frac{d\theta}{(1 + e \cos \theta)^n},$$

where  $n$  is a positive integer, may be reduced to an integrable form. For

$$\int \frac{d\theta}{(1 + e \cos \theta)^n} = \int \frac{d\theta}{1 + e \cos \theta} \frac{1}{(1 + e \cos \theta)^{n-1}};$$

hence, by equations (5) and (7),

$$\int_0 \frac{d\theta}{(1 + e \cos \theta)^n} = \frac{1}{(1 - e^2)^{n-\frac{1}{2}}} \int_0 (1 - e \cos \phi)^{n-1} d\phi.$$

By expanding  $(1 - e \cos \phi)^{n-1}$ , the last expression is reduced to a series of integrals involving powers of  $\cos \phi$ ; these may be evaluated by the methods given in this section and Section VI, and the results expressed in terms of  $\theta$  by means of equation (2) or of equation (7).

### Examples III.

$$1. \int \tan^4 mx \, dx, \quad \frac{\tan^3 mx}{3m} - \frac{\tan mx}{m} + x.$$

$$2. \int_0^{\frac{\pi}{4}} \tan^7 x \, dx, \quad \frac{5}{12} - \frac{1}{2} \log 2.$$

$$3. \int \sec^4 (\theta + \alpha) \, d\theta, \quad \frac{\tan^3 (\theta + \alpha)}{3} + \tan (\theta + \alpha)$$

$$4. \int_0^{\frac{\pi}{m}} \sin^3 mx \, dx, \quad \frac{4}{3m}.$$

$$5. \int \sin^3 \theta \cos^3 \theta \, d\theta, \quad \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5}.$$

$$6. \int \sqrt{(\sin \theta)} \cos^5 \theta \, d\theta, \quad \frac{2}{3} \sin^{\frac{3}{2}} \theta - \frac{4}{7} \sin^{\frac{7}{2}} \theta + \frac{2}{11} \sin^{\frac{11}{2}} \theta.$$

$$7. \int_0^{\frac{\pi}{2}} \cos^4 \theta \sin^3 \theta \, d\theta, \quad \frac{2}{35}.$$

$$8. \int \frac{\sin^3 \theta \, d\theta}{\sqrt{(\cos \theta)}}, \quad \frac{2}{5} \cos^{\frac{5}{2}} \theta - 2 \cos^{\frac{1}{2}} \theta.$$

$$9. \int \frac{d\theta}{\sin^3 \theta \cos^3 \theta}, \quad \text{Multiply by } \sin^2 \theta + \cos^2 \theta. \quad \tan \theta - \cot \theta.$$

$$10. \int \frac{\sin^3 x}{\cos^5 x} \, dx, \quad \text{See Art. 28.} \quad \frac{\tan^4 x}{4}.$$

$$11. \int \frac{d\theta}{\sin^3 \theta \cos^3 \theta}, \quad \frac{1}{2} (\tan^2 \theta - \cot^2 \theta) + 2 \log \tan \theta.$$

$$12. \int \frac{\sqrt{(\sin \theta)} \, d\theta}{\cos^{\frac{5}{2}} \theta}, \quad \frac{2}{3} \tan^{\frac{3}{2}} \theta.$$

$$13. \int \frac{\sin^3 x \, dx}{\cos^6 x}, \quad \frac{1}{5 \cos^5 x} - \frac{1}{3 \cos^3 x}.$$

$$14. \int \frac{\sin^2 x \, dx}{\cos^6 x}, \quad \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3}.$$

$$15. \int \sin^2 \theta \cos^2 \theta \, d\theta, \quad \frac{1}{16} [2\theta - \sin 2\theta \cos 2\theta]$$



$$16. \int_0^{\frac{\pi}{m}} \sin^2 mx \, dx, \quad \frac{\pi}{2m}.$$

$$17. \int \frac{\sin^2 \theta \, d\theta}{\cos \theta}, \quad \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right] - \sin \theta.$$

$$18. \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 \theta \, d\theta}{\sin \theta}, \quad \frac{1}{2} (\log 3 - 1).$$

$$19. \int \frac{d\theta}{\sin \theta + \cos \theta}, \quad \frac{1}{\sqrt{2}} \log \tan \left[ \frac{\theta}{2} + \frac{\pi}{8} \right].$$

$$20. \int \frac{dx}{1 + \cos x}, \quad \tan \frac{1}{2}x.$$

$$21. \int_{\frac{\pi}{2}}^{\pi} \frac{dx}{1 - \cos x}, \quad 1 - \cot \frac{1}{2}x.$$

$$22. \int \frac{dx}{1 \pm \sin x},$$

Multiply both terms of the fraction by  $1 \mp \sin x$ .  $\tan x \mp \sec x$ .

$$23. \int \frac{d\theta}{\sec \theta \pm \tan \theta}, \quad \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right] \pm \log \cos \theta.$$

$$24. \int \cos \theta \cos 3\theta \, d\theta. \quad \text{See Art. 33.} \quad \frac{1}{8} \sin 4\theta + \frac{1}{4} \sin 2\theta.$$

$$25. \int_0^{\frac{\pi}{2}} \cos \theta \cos 2\theta \, d\theta, \quad \frac{1}{3}.$$

$$26. \int_0^{\frac{\pi}{4}} \sin^2 \theta \sin 2\theta \, d\theta, \quad \frac{1}{2} \sin^4 \theta \Big|_0^{\frac{\pi}{4}} = \frac{1}{8}.$$

$$27. \int_0^{\frac{\pi}{2}} \sin 3\theta \sin 2\theta \, d\theta, \quad \frac{2}{5}.$$

$$28. \int_0^{\theta} \sin m\theta \cos n\theta d\theta,$$

$$\frac{1 - \cos(m+n)\theta}{2(m+n)} + \frac{1 - \cos(m-n)\theta}{2(m-n)}.$$

$$29. \int \cos x \cos 2x \cos 3x dx,$$

Reduce products to sums by means of equation (2), Art. 33.

$$\frac{1}{4} \left[ \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + x \right].$$

$$30. \int_0^{\pi} \sqrt{1 - \cos x} dx, \quad 2\sqrt{2}.$$

$$31. \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}, \quad \frac{1}{ab} \tan^{-1} \left[ \frac{b}{a} \tan x \right].$$

$$32. \int \frac{dx}{1 + \cos^2 x}, \quad \frac{1}{\sqrt{2}} \tan^{-1} \frac{\tan x}{\sqrt{2}}.$$

$$33. \int \frac{dx}{a^2 \cos^2 x - b^2 \sin^2 x}, \quad \frac{1}{2ab} \log \frac{a + b \tan x}{a - b \tan x}.$$

$$34. \int \frac{\sin x dx}{\sqrt{3 \cos^2 x + 4 \sin^2 x}}, \quad \cos^{-1} \left\{ \frac{1}{2} \cos x \right\}.$$

$$35. \int \frac{\sin x \cos^2 x dx}{1 + a^2 \cos^2 x},$$

Putting  $y$  for  $\cos x$ , the integral becomes  $-\int \frac{y^2 dy}{1 + a^2 y^2}.$

$$-\frac{\cos x}{a^2} + \frac{\tan^{-1}(a \cos x)}{a^3}.$$

$$36. \int \frac{d\theta}{a + b \sin \theta}.$$

Put  $\sin \theta = \cos (\theta - \frac{1}{2}\pi)$ , and use formulas (G) and (G').

$$\text{If } a > b, \frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left[ \sqrt{\frac{a-b}{a+b}} \tan \frac{2\theta - \pi}{4} \right].$$

$$\text{If } a < b, \frac{1}{\sqrt{(b^2 - a^2)}} \log \frac{\sqrt{(b+a)} + \sqrt{(b-a)} \tan (\frac{1}{2}\theta - \frac{1}{4}\pi)}{\sqrt{(b+a)} - \sqrt{(b-a)} \tan (\frac{1}{2}\theta - \frac{1}{4}\pi)}.$$

$$37. \int \frac{d\theta}{3 + 5 \cos \theta}, \quad \frac{1}{4} \log \frac{2 + \tan \frac{1}{2}\theta}{2 - \tan \frac{1}{2}\theta}.$$

$$38. \int \frac{d\theta}{5 + 3 \cos \theta}, \quad \frac{1}{2} \tan^{-1} \left[ \frac{1}{2} \tan \frac{1}{2}\theta \right].$$

$$39. \int \frac{d\theta}{5 - 4 \cos \theta}, \quad \frac{2}{3} \tan^{-1} \left\{ 3 \tan \frac{1}{2}\theta \right\}.$$

$$40. \int \frac{d\theta}{2 \cos \theta - 1}, \quad \frac{1}{\sqrt{3}} \log \frac{1 + \sqrt{3} \tan \frac{1}{2}\theta}{1 - \sqrt{3} \tan \frac{1}{2}\theta}.$$

$$41. \int_0^{\frac{\pi}{2}} \frac{d\theta}{3 - \cos \theta}, \quad \frac{\tan^{-1} \sqrt{2}}{\sqrt{2}}.$$

$$42. \int_0^{\frac{\pi}{3}} \frac{d\theta}{2 - \cos \theta}, \quad \frac{\pi}{2\sqrt{3}}.$$

$$43. \int \frac{d\theta}{(1 + e \cos \theta)^2}, \quad \text{See Art. 36.}$$

$$\frac{1}{(1 - e^2)^{\frac{3}{2}}} \cos^{-1} \frac{e + \cos \theta}{1 + e \cos \theta} - \frac{e}{1 - e^2} \frac{\sin \theta}{1 + e \cos \theta}.$$

$$44. \int_0^{\pi} \frac{d\theta}{(1 + e \cos \theta)^3}, \quad \frac{(2 + e^2) \pi}{2(1 - e^2)^{\frac{5}{2}}}.$$

$$45. \int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx,$$

*Solution :—*

By adding and subtracting an undetermined constant, the fraction may be written in the form

$$\frac{p \cos x + q \sin x + A(a \cos x + b \sin x)}{a \cos x + b \sin x} - A,$$

we may now assume

$$p \cos x + q \sin x + A(a \cos x + b \sin x) = k(b \cos x - a \sin x);$$

the expression is then readily integrated, and  $A$  and  $k$  so determined as to make the equation last written an identity. The result is

$$\int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx = \frac{ap + bq}{a^2 + b^2} x + \frac{bp - aq}{a^2 + b^2} \log(a \cos x + b \sin x).$$

$$46. \int \frac{dx}{a + b \tan x}, \quad \text{See Ex. 45.}$$

$$\frac{ax}{a^2 + b^2} + \frac{b}{a^2 + b^2} \log(a \cos x + b \sin x).$$

47. Find the area of the ellipse

$$x = a \cos \phi$$

$$y = b \sin \phi.$$

$$-4ab \int_{\frac{1}{2}\pi}^0 \sin^2 \phi d\phi = \pi ab.$$

48. Find the area of the cycloid

$$x = a(\psi - \sin \psi)$$

$$y = a(1 - \cos \psi).$$

$$a^2 \int_0^{2\pi} (1 - \cos \psi)^2 d\psi = 3a^2\pi.$$

49. Find the area of the trochoid ( $b < a$ )

$$x = a\psi - b \sin \psi \qquad y = a - b \cos \psi.$$

$$(2a^2 + b^2) \pi.$$

50. Find the area of the loop, and also the area between the curve and the asymptote, in the case of the strophoid whose polar equation is

$$r = a (\sec \theta \pm \tan \theta).$$

*Solution :—*

Using  $\theta$  as an auxiliary variable, we have

$$x = a (1 \pm \sin \theta) \qquad y = a \left[ \tan \theta \pm \frac{\sin^2 \theta}{\cos \theta} \right],$$

the upper sign corresponding to the infinite branch, and the lower to the loop. Hence, for the half areas we obtain

$$+ a^2 \int_0^{\frac{1}{2}\pi} \sin \theta \, d\theta + a^2 \int_0^{\frac{1}{2}\pi} \sin^2 \theta \, d\theta = a^2 \left[ 1 + \frac{\pi}{4} \right]$$

and

$$- a^2 \int_{\frac{1}{2}\pi}^0 \sin \theta \, d\theta + a^2 \int_{\frac{1}{2}\pi}^0 \sin^2 \theta \, d\theta = a^2 \left[ 1 - \frac{\pi}{4} \right].$$

## CHAPTER II.

## METHODS OF INTEGRATION—CONTINUED.

## IV.

*Integration by Change of Independent Variable.*

**37.** IF  $x$  is the independent variable used in expressing an integral, and  $y$  is any function of  $x$ , the integral may be expressed in terms of  $y$ , by substituting for  $x$  and  $dx$  their values in terms of  $y$  and  $dy$ . By properly assuming the function  $y$ , the integral may frequently be made to take a directly integrable form. For example, the integral

$$\int \frac{x \, dx}{(ax + b)^2}$$

will obviously be simplified by assuming

$$y = ax + b$$

for the new independent variable. This assumption gives

$$x = \frac{y - b}{a}, \quad \text{whence} \quad dx = \frac{dy}{a};$$

substituting, we have

$$\begin{aligned} \int \frac{x \, dx}{(ax + b)^2} &= \frac{1}{a^2} \int \frac{(y - b) \, dy}{y^2} \\ &= \frac{1}{a^2} \log y + \frac{b}{a^2 y}; \end{aligned}$$

or replacing  $y$  by  $x$  in the result,

$$\int \frac{x \, dx}{(ax + b)^2} = \frac{1}{a^2} \log (ax + b) + \frac{b}{a^2 (ax + b)}.$$

**38.** Again, if in the integral

$$\int \frac{dx}{\varepsilon^x - 1}$$

we put  $y = \varepsilon^x$ , whence

$$x = \log y,$$

and

$$dx = \frac{dy}{y},$$

we have

$$\int \frac{dx}{\varepsilon^x - 1} = \int \frac{dy}{y(y - 1)}.$$

Hence, by formula (A), Art. 17,

$$\int \frac{dx}{\varepsilon^x - 1} = \log \frac{y - 1}{y} = \log (\varepsilon^x - 1) - x.$$

It is easily seen that, by this change of independent variable, any integral in which the coefficient of  $dx$  is a rational function of  $\varepsilon^x$ , may be transformed into one in which the coefficient of  $dy$  is a rational function of  $y$ .

### *Transformation of Trigonometric Forms.*

**39.** When in a trigonometric integral the coefficient of  $d\theta$  is a rational function of  $\tan \theta$ , the integral will take a rational algebraic form if we put

$$\tan \theta = x,$$

whence

$$d\theta = \frac{dx}{1 + x^2}.$$

For example, by this transformation, we have

$$\int \frac{d\theta}{1 + \tan \theta} = \int \frac{dx}{(1 + x^2)(1 + x)}.$$

Decomposing the fraction in the latter integral, we have

$$\begin{aligned} \int \frac{d\theta}{1 + \tan \theta} &= \frac{1}{2} \int \frac{dx}{1 + x^2} - \frac{1}{2} \int \frac{x dx}{1 + x^2} + \frac{1}{2} \int \frac{dx}{1 + x} \\ &= \frac{1}{2} \tan^{-1} x - \frac{1}{4} \log (1 + x^2) + \frac{1}{2} \log (1 + x) \\ &= \frac{1}{2} \left[ \theta + \log \frac{1 + \tan \theta}{\sec \theta} \right], \end{aligned}$$

or 
$$\int \frac{d\theta}{1 + \tan \theta} = \frac{1}{2} [\theta + \log (\cos \theta + \sin \theta)].$$

**40.** The method given in the preceding article may be employed when the coefficient of  $d\theta$  is a *homogeneous rational function* of  $\sin \theta$  and  $\cos \theta$ , of a degree indicated by an even integer; for such a function is a rational function of  $\tan \theta$ . It may also be noticed that, when the coefficient of  $d\theta$  is *any rational function* of  $\sin \theta$  and  $\cos \theta$ , the integral becomes rational and algebraic if we put

$$z = \tan \frac{\theta}{2};$$

for this gives

$$\sin \theta = \frac{2z}{1 + z^2}, \quad \cos \theta = \frac{1 - z^2}{1 + z^2}, \quad d\theta = \frac{2dz}{1 + z^2}.$$

This transformation has in fact been already employed in the integration of  $\frac{d\theta}{a + b \cos \theta}$ . See Art. 35.



*Limits of the Transformed Integral.*

**41.** When a definite integral is transformed by a change of independent variable, it is necessary to make a corresponding change in the limits. If, for example, in the integral

$$\int_a^{\infty} \frac{dx}{(a^2 + x^2)^2}$$

we put  $x = a \tan \theta$ , whence  $dx = a \sec^2 \theta d\theta$ ,

we must at the same time replace the limits  $a$  and  $\infty$ , which are values of  $x$ , by  $\frac{1}{4}\pi$  and  $\frac{1}{2}\pi$ , the corresponding values of  $\theta$ . Thus

$$\begin{aligned} \int_a^{\infty} \frac{dx}{(a^2 + x^2)^2} &= \frac{1}{a^3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= \frac{1}{2a^3} \left[ \theta + \sin \theta \cos \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi - 2}{8a^3}. \end{aligned}$$

*The Reciprocal of  $x$  taken as the New Independent Variable.*

**42.** In the case of fractional integrals, it is sometimes useful to take the reciprocal of  $x$  as the new independent variable. For example, let the given integral be

$$\int \frac{dx}{x^3(x+1)^2}.$$

Putting  $x = \frac{1}{y}$ , whence  $dx = -\frac{dy}{y^2}$ ,

we have

$$\int \frac{dx}{x^3(x+1)^2} = - \int \frac{y^3 dy}{y^2 \left(1 + \frac{1}{y}\right)^2} = - \int \frac{y^3 dy}{(y+1)^3}.$$

Transforming again by putting  $z = y + 1$ , the integral becomes

$$\begin{aligned} - \int \frac{(z-1)^3}{z^2} dz &= - \int z dz + 3 \int dz - 3 \int \frac{dz}{z} + \int \frac{dz}{z^2} \\ &= -\frac{z^2}{2} + 3z - 3 \log z - \frac{1}{z}. \end{aligned}$$

Therefore, since  $z = y + 1 = \frac{1}{x} + 1 = \frac{x+1}{x}$ ,

$$\int \frac{dx}{x^3(x+1)^2} = -\frac{1}{2x^2} + \frac{2}{x} + \frac{1}{x+1} - 3 \log \frac{x+1}{x} + C.$$

**43.** In the above example we see that the single substitution  $z = \frac{x+1}{x}$  is equivalent to the two substitutions which are suggested in the process. In like manner, in the case of the more general integral

$$\int \frac{dx}{(x-a)^m(x-b)^n},$$

the successive transformations which suggest themselves are found to be equivalent to a single one in which the new variable is

$$z = \frac{x-a}{x-b}, \quad \text{whence} \quad x = \frac{bz-a}{z-1}.$$

The result is

$$\frac{1}{(a-b)^{m+n-1}} \int \frac{(1-z)^{m+n-2}}{z^m} dz,$$

which, when  $m + n - 2$  is a positive integer, is directly integrable after expansion by the Binomial Theorem.

*A Power of  $x$  taken as the New Independent Variable.*

**44.** The transformation of an integral by the assumption

$$y = x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

is not generally useful, since the substitution

$$x = y^{\frac{1}{n}}, \quad \text{whence} \quad dx = \frac{1}{n} y^{\frac{1}{n}-1} dy,$$

will usually introduce radicals. Exceptional cases, however, occur. For, since logarithmic differentiation of equation (I) gives

$$\frac{dx}{x} = \frac{dy}{ny}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

it is evident that, if the expression to be integrated is the product of  $\frac{dx}{x}$  and a function of  $x^n$ , the transformed expression will be the product of  $\frac{dy}{ny}$  and the like function of  $y$ .

For example, the substitution  $y = x^4$  transforms

$$\frac{(x^4 - 1)dx}{x(x^4 + 1)} \quad \text{into} \quad \frac{(y - 1)dy}{4y(y + 1)}.$$

Hence, decomposing the fraction in the latter expression,

$$\int \frac{(x^4 - 1)dx}{x(x^4 + 1)} = \frac{1}{4} \log \frac{(y + 1)^2}{y} = \log \frac{\sqrt[4]{(x^4 + 1)}}{x}.$$

Again, putting  $x^3 = y^2$  or  $y = x^{\frac{3}{2}}$ , we have

$$\int \frac{dx}{x \sqrt[4]{(x^3 - a^3)}} = \frac{2}{3} \int \frac{dy}{y \sqrt[4]{(y^2 - a^3)}} = \frac{2}{3a^{\frac{3}{2}}} \sec^{-1} \frac{y}{a^{\frac{3}{2}}} = \frac{2}{3a^{\frac{3}{2}}} \sec^{-1} \left( \frac{x}{a} \right)^{\frac{3}{2}}.$$

**45.** When this method is applied to an integral whose form at the same time suggests the employment of the reciprocal, as in Art. 42, we may at once assume  $y = x^{-n}$ . Thus, given the integral

$$\int_1^{\infty} \frac{dx}{x^4(2 + x^3)};$$

putting  $y = x^{-3}$ ,                      whence  $\frac{dx}{x} = -\frac{dy}{3y}$

we obtain

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^4(2 + x^3)} &= -\frac{1}{3} \int_1^0 \frac{y \, dy}{2y + 1} \\ &= -\left[ \frac{y}{6} + \frac{\log(2y + 1)}{12} \right]_1^0 = \frac{2 - \log 3}{12}. \end{aligned}$$

#### Examples IV.

1.  $\int \frac{1+x}{(2+x)^2} dx,$

$$\log(2+x) + \frac{1}{2+x}$$

$$2. \int \frac{x \, dx}{(1-x)^3}, \quad \frac{2x-1}{2(1-x)^2}.$$

$$3. \int \frac{x^2 - x + 1}{(2x+1)^2} \, dx, \quad \frac{2x+1}{8} - \frac{\log(2x+1)}{2} - \frac{7}{8(2x+1)}.$$

$$4. \int_{-1}^0 \frac{x^2 \, dx}{(x+2)^3}, \quad \log y + \frac{4y-2}{y^2} \Big|_1^2 = \log 2 - \frac{1}{2}.$$

$$5. \int \frac{dx}{1+\varepsilon^x}, \quad x - \log(1+\varepsilon^x).$$

$$6. \int \frac{dx}{\varepsilon^x - \varepsilon^{-x}}, \quad \frac{1}{2} \log \frac{\varepsilon^x - 1}{\varepsilon^x + 1}.$$

$$7. \int_{-\infty}^0 \frac{\varepsilon^{2x} \, dx}{\varepsilon^x + 1}, \quad 1 - \log 2.$$

$$8. \int \frac{\varepsilon^x + 1}{1 - \varepsilon^{-x}} \, dx, \quad \varepsilon^x + 2 \log(\varepsilon^x - 1).$$

$$9. \int \frac{2 + \tan \theta}{3 - \tan \theta} \, d\theta, \quad \frac{\theta - \log(3 \cos \theta - \sin \theta)}{2}.$$

$$10. \int \frac{d\theta}{\tan^2 \theta - 1}, \quad \frac{1}{4} \log \frac{\tan \theta - 1}{\tan \theta + 1} - \frac{\theta}{2}.$$

$$11. \int \frac{\tan^2 \theta \, d\theta}{\tan^2 \theta - 1}, \quad \frac{1}{4} \log \frac{\tan \theta - 1}{\tan \theta + 1} + \frac{\theta}{2}.$$

$$12. \int \frac{\cos \theta \, d\theta}{a \cos \theta - b \sin \theta}, \quad \frac{a\theta - b \log(a \cos \theta - b \sin \theta)}{a^2 + b^2}.$$

$$13. \int \frac{\cos \theta \, d\theta}{\cos (\alpha + \theta)}, \quad \text{Put } \theta' = \alpha + \theta.$$

$$(\theta + \alpha) \cos \alpha - \sin \alpha \log \cos (\theta + \alpha).$$

$$14. \int \frac{\sin (\theta + \alpha)}{\sin (\theta + \beta)} \, d\theta,$$

$$(\theta + \beta) \cos (\alpha - \beta) + \sin (\alpha - \beta) \log \sin (\theta + \beta).$$

$$15. \int \tan (\theta + \alpha) \cos \theta \, d\theta, \quad -\cos \theta + \sin \alpha \log \tan \frac{2\theta + 2\alpha + \pi}{4}$$

$$16. \int_0^\alpha \frac{\cos \theta \, d\theta}{\sin (\alpha + \theta)}, \quad \cos \alpha \log (2 \cos \alpha) + \alpha \sin \alpha.$$

$$17. \int_0^{\frac{\pi}{3}} \frac{\cos \frac{1}{2} \theta}{\cos \theta} \, d\theta, \quad \frac{1}{\sqrt{2}} \log \frac{\sqrt{2} + 2 \sin \theta'}{\sqrt{2} - 2 \sin \theta'} \bigg|_0^{\frac{\pi}{6}} = \frac{\log (3 + 2 \sqrt{2})}{\sqrt{2}}.$$

$$18. \int \frac{\sin \frac{1}{2} \theta \, d\theta}{\sin \theta}, \quad \log \tan \frac{\pi + \theta}{4}.$$

$$19. \int \frac{x^3 \, dx}{(a^2 + x^2)^2}, \quad \frac{1}{2} \log (a^2 + x^2) + \frac{a^2}{2(a^2 + x^2)}.$$

$$20. \int \frac{dx}{x^3 (1 + x^2)}, \quad \log \frac{\sqrt{1 + x^2}}{x} - \frac{1}{2x^2}.$$

$$21. \int_0^\infty \frac{x^2 \, dx}{(1 + x^2)^3}, \quad \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \frac{\pi}{16}.$$

$$22. \int_1^\infty \frac{dx}{x^3 (1 + x^2)^2}, \quad \frac{3}{4} - \log 2$$

$$23. \int \frac{dx}{x^3(x+1)}, \quad -\frac{1}{2x^2} + \frac{1}{x} - \log \frac{x+1}{x}.$$

$$24. \int \frac{dx}{(1-x)^3 x}, \quad \frac{1}{2(1-x)^2} + \frac{1}{1-x} + \log \frac{x}{1-x}.$$

$$25. \int_1^\infty \frac{dx}{x^3(x^2+2)}, \quad -\frac{y^2}{4} + \frac{1}{8} \log(2y^2+1) \Big|_1^\infty = \frac{2 - \log 3}{8}.$$

$$26. \int \frac{dx}{x(a+bx^4)}, \quad \frac{1}{4a} \log \frac{x^4}{a+bx^4}.$$

$$27. \int \frac{dx}{x(x^n+a^n)}, \quad \frac{1}{na^n} \log \frac{x^n}{x^n+a^n}.$$

$$28. \int \frac{(x^3+1)dx}{x(x^3-1)}, \quad \frac{2}{3} \log(x^3-1) - \log x.$$

$$29. \int \frac{dx}{x\sqrt{x^n-a^n}}, \quad \frac{2}{na^{\frac{n}{2}}} \sec^{-1} \left( \frac{x}{a} \right)^{\frac{n}{2}}.$$

## V.

*Integrals Containing Radicals.*

**46.** An integral containing a single radical, in which the expression under the radical sign is of the first degree, is *rationalized*, that is, transformed into a rational integral, by taking the radical as the value of the new independent variable. Thus, given the integral

$$\int \frac{dx}{1+\sqrt{x+1}},$$

putting  $y = \sqrt{x+1}$ ,

whence  $x = y^2 - 1$ , and  $dx = 2y dy$ ,

we have

$$\begin{aligned} \int \frac{dx}{1 + \sqrt{x+1}} &= 2 \int \frac{y dy}{1+y} = 2 \int dy - 2 \int \frac{dy}{1+y} \\ &= 2y - 2 \log(1+y) \\ &= 2\sqrt{x+1} - 2 \log[1 + \sqrt{x+1}]. \end{aligned}$$

**47.** The same method evidently applies whenever all the radicals which occur in the integral are powers of a single radical, in which the expression under the radical sign is linear. Thus, in the integral

$$\int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}} + (x-1)^{\frac{1}{2}}},$$

the radicals are powers of  $(x-1)^{\frac{1}{6}}$ ; hence we put  $y = (x-1)^{\frac{1}{6}}$ , and obtain

$$\begin{aligned} \int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}} + (x-1)^{\frac{1}{2}}} &= 6 \int_0^1 \frac{y^5 dy}{y^4 + y^3} \\ &= 6 \int_0^1 (y-1) dy + 6 \int_0^1 \frac{dy}{y+1} = -3 + 6 \log 2. \end{aligned}$$

**48.** An integral in which a binomial expression occurs under the radical sign can sometimes be reduced to the form considered above by the method of Art. 44. For example, since

$$\int \frac{dx}{x(x^3+1)^{\frac{1}{4}}}$$



fulfils the condition given in Art. 44 when  $n = 3$ , the quantity under the radical sign may be reduced to the first degree. Hence, in accordance with Art. 46, we may take the radical as the value of the new independent variable. Thus, putting

$$z = (x^3 + 1)^{\frac{1}{4}},$$

whence  $x^3 = z^4 - 1$ , and  $\frac{dx}{x} = \frac{4z^3 dz}{z^4 - 1}$ ,

we have

$$\int \frac{dx}{x(x^3 + 1)^{\frac{1}{4}}} = \frac{4}{3} \int \frac{z^2 dz}{z^4 - 1}.$$

Decomposing the fraction in the latter integral (see Art. 22), we have finally

$$\int \frac{dx}{x(x^3 + 1)^{\frac{1}{4}}} = \frac{2}{3} \tan^{-1} \left[ (x^3 + 1)^{\frac{1}{4}} \right] + \frac{1}{3} \log \frac{(x^3 + 1)^{\frac{1}{4}} - 1}{(x^3 + 1)^{\frac{1}{4}} + 1}.$$

*Radicals of the Form  $\sqrt{(ax^2 + b)}$ .*

**49.** It is evident that the method given in the preceding article is applicable to all integrals of the general form

$$\int x^{2m+1} (ax^2 + b)^{n+\frac{1}{2}} dx, \quad . \quad . \quad . \quad . \quad (I)$$

in which  $m$  and  $n$  are positive or negative integers. These integrals are therefore rationalized by putting

$$y = \sqrt{(ax^2 + b)}.$$

Putting  $m = 0$ , the form (1) includes the directly integrable case

$$\int (ax^2 + b)^{n+\frac{1}{2}} x dx.$$

50. As an illustration let us take the integral

$$\int \frac{dx}{x \sqrt{x^2 + a^2}};$$

putting

$$y = \sqrt{x^2 + a^2},$$

whence

$$x^2 = y^2 - a^2,$$

and

$$\frac{dx}{x} = \frac{y dy}{y^2 - a^2},$$

we have

$$\int \frac{dx}{x \sqrt{x^2 + a^2}} = \int \frac{dy}{y^2 - a^2}.$$

Hence, by equation ( $A'$ ) Art. 17,

$$\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{2a} \log \frac{y - a}{y + a} = \frac{1}{2a} \log \frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a}.$$

Rationalizing the denominator of the fraction in this result, we have

$$\frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a} = \frac{[\sqrt{x^2 + a^2} - a]^2}{x^2}.$$

Therefore

$$\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \log \frac{\sqrt{x^2 + a^2} - a}{x} . . . . (H)$$

In a similar manner we may prove that

$$\int \frac{dx}{x \sqrt{(a^2 - x^2)}} = \frac{1}{a} \log \frac{a - \sqrt{(a^2 - x^2)}}{x} \cdot \cdot \cdot (I)$$

51. Integrals of the form

$$\int x^{2m} (ax^2 + b)^{n+\frac{1}{2}} dx \cdot \cdot \cdot (2)$$

are reducible to the form (1) Art. 49, by first putting  $y = \frac{1}{x}$ .

For example:

$$\int \frac{dx}{(ax^2 + b)^{\frac{3}{2}}}$$

is of the form (2); but, putting  $x = \frac{1}{y}$ , whence

$$\sqrt{(ax^2 + b)} = \frac{\sqrt{(a + by^2)}}{y} \quad \text{and} \quad dx = -\frac{dy}{y^2},$$

we obtain

$$\int \frac{dx}{(ax^2 + b)^{\frac{3}{2}}} = - \int \frac{y dy}{(a + by^2)^{\frac{3}{2}}}.$$

The resulting expression is in this case directly integrable. Thus

$$\int \frac{dx}{(ax^2 + b)^{\frac{3}{2}}} = \frac{1}{b \sqrt{(a + by^2)}} = \frac{x}{b \sqrt{(ax^2 + b)}} \cdot \cdot \cdot (J)$$

# *Integration of $\frac{dx}{\sqrt{(x^2 \pm a^2)}}.$*

**52.** If we assume a new variable  $z$  connected with  $x$  by the relation

$$z - x = \sqrt{(x^2 \pm a^2)}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have, by squaring,

$$z^2 - 2zx = \pm a^2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, by differentiating this equation,

$$2(z - x) dz - 2z dx = 0;$$

whence

$$\frac{dx}{z - x} = \frac{dz}{z},$$

or by equation (1),

$$\frac{dx}{\sqrt{(x^2 \pm a^2)}} = \frac{dz}{z}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating equation (3), we obtain

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log z = \log [x + \sqrt{(x^2 \pm a^2)}]. \quad . \quad . \quad (K)$$

**53.** Since the value of  $x$  in terms of  $z$ , derived from equation (2) of the preceding article, is rational, it is obvious that this transformation may be employed to rationalize any expression which consists of the product of  $\frac{dx}{\sqrt{(x^2 \pm a^2)}}$  and a rational function of  $x$ . For example, let us find the value of

$$\int \sqrt{(x^2 \pm a^2)} dx,$$

which may be written in the form

$$\int (x^2 \pm a^2) \frac{dx}{\sqrt{x^2 \pm a^2}}.$$

By equation (2)

$$x = \frac{z^2 \mp a^2}{2z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence

$$x^2 \pm a^2 = \frac{(z^2 \pm a^2)^2}{4z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Therefore, by equations (3) and (5),

$$\begin{aligned} \int \sqrt{x^2 \pm a^2} \, dx &= \frac{1}{4} \int \frac{(z^2 \pm a^2)^2}{z^3} \, dz \\ &= \frac{1}{4} \int z \, dz \pm \frac{a^2}{2} \int \frac{dz}{z} + \frac{a^4}{4} \int \frac{dz}{z^3} \\ &= \frac{z^4 - a^4}{8z^2} \pm \frac{a^2}{2} \log z. \end{aligned}$$

By equations (4) and (5), the first term of the last member is equal to  $\frac{1}{2} x \sqrt{x^2 \pm a^2}$ . Hence

$$\int \sqrt{x^2 \pm a^2} \, dx = \frac{x \sqrt{x^2 \pm a^2}}{2} \pm \frac{a^2}{2} \log [x + \sqrt{x^2 \pm a^2}]. \quad . \quad (L)$$

### *Transformation to Trigonometric Forms.*

#### **54.** Integrals involving either of the radicals

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \text{or} \quad \sqrt{x^2 - a^2}$$

can be transformed into rational trigonometric integrals. The transformation is effected in the first case by putting

$$x = a \sin \theta, \quad \text{whence} \quad \sqrt{a^2 - x^2} = a \cos \theta;$$

in the second case, by putting

$$x = a \tan \theta, \quad \text{whence} \quad \sqrt{a^2 + x^2} = a \sec \theta;$$

and in the third case, by putting

$$x = a \sec \theta, \quad \text{whence} \quad \sqrt{x^2 - a^2} = a \tan \theta.$$

55. As an illustration, let us take the integral

$$\int \sqrt{a^2 - x^2} \, dx;$$

putting  $x = a \sin \theta$ , we have  $\sqrt{a^2 - x^2} = a \cos \theta$ ,  $dx = a \cos \theta \, d\theta$ ; hence

$$\begin{aligned} \int \sqrt{a^2 - x^2} \, dx &= a^2 \int \cos^2 \theta \, d\theta \\ &= \frac{a^2 \theta}{2} + \frac{a^2 \sin \theta \cos \theta}{2}, \end{aligned}$$

by formula (C) Art. 29. Replacing  $\theta$  by  $x$  in the result,

$$\int \sqrt{a^2 - x^2} \, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x \sqrt{a^2 - x^2}}{2}. \quad (M)$$

Regarding the radical as a positive quantity, the value of  $\theta$  may be restricted to the *primary* value of the symbol  $\sin^{-1} \frac{x}{a}$  (see Diff. Calc., Art. 54); that is, as  $x$  passes from  $-a$  to  $+a$ ,  $\theta$  passes from  $-\frac{1}{2}\pi$  to  $+\frac{1}{2}\pi$ .

*Radicals of the Form  $\sqrt{ax^2 + bx + c}$ .*

**56.** When a radical of the form  $\sqrt{ax^2 + bx + c}$  occurs in an integral, a simple change of independent variable will cause the radical to assume one of the forms considered in the preceding articles. Thus, if the coefficient of  $x^2$  is positive,

$$\sqrt{ax^2 + bx + c} = \sqrt{a} \sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}},$$

in which, if we put  $x + \frac{b}{2a} = y$ , the radical takes the form  $\sqrt{y^2 + a^2}$  or  $\sqrt{y^2 - a^2}$ , according as  $4ac - b^2$  is positive or negative. If  $a$  is negative, the radical can in like manner be reduced to the form  $\sqrt{a^2 - y^2}$  or  $\sqrt{-a^2 - y^2}$ ; but the latter will never occur, since it is imaginary for all values of  $y$ , and therefore imaginary for all values of  $x$ .

For example, by this transformation, the integral

$$\int \frac{dx}{(ax^2 + bx + c)^{\frac{3}{2}}}$$

can be reduced at once to the form (F), Art. 51. Thus

$$\begin{aligned} \int \frac{dx}{(ax^2 + bx + c)^{\frac{3}{2}}} &= \int \frac{dx}{\left[a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}\right]^{\frac{3}{2}}} \\ &= \frac{x + \frac{b}{2a}}{\frac{4ac - b^2}{4a} \sqrt{ax^2 + bx + c}} = \frac{4ax + 2b}{(4ac - b^2) \sqrt{ax^2 + bx + c}}. \end{aligned}$$

**57.** When the form of the integral suggests a further change of independent variable, we may at once assume the expression for the new variable in the required form. For example, given the integral

$$\int \sqrt[3]{2ax - x^2} x dx;$$

we have  $\sqrt[3]{2ax - x^2} = \sqrt[3]{a^2 - (x - a)^2}$

hence (see Art. 54), if we put  $x - a = a \sin \theta$ , we have

$$\sqrt[3]{2ax - x^2} = a \cos \theta,$$

$$x = a(1 + \sin \theta), \quad dx = a \cos \theta d\theta;$$

$$\begin{aligned} \therefore \int \sqrt[3]{2ax - x^2} x dx &= a^3 \int \cos^2 \theta (1 + \sin \theta) d\theta \\ &= \frac{a^3}{2} (\theta + \sin \theta \cos \theta) - \frac{a^3}{3} \cos^3 \theta \\ &= \frac{a^3}{2} \sin^{-1} \frac{x-a}{a} + \frac{a}{2} (x-a) \sqrt[3]{2ax - x^2} - \frac{1}{3} (2ax - x^2)^{\frac{3}{2}} \\ &= \frac{a^3}{2} \sin^{-1} \frac{x-a}{a} + \frac{1}{6} \sqrt[3]{2ax - x^2} [2x^2 - ax - 3a^2]. \end{aligned}$$

### *The Integrals*

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} \quad \text{and} \quad \int \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}}.$$

**58.** An integral of the form  $\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$  may by the method of Art. 56, be reduced to the form (K), Art. 52, or to the form (j'), Art. 10, according as  $a$  is positive or negative.



But, when the integral appears in one of the above forms (the quadratic under the radical sign admitting of linear factors), another mode of transforming is often convenient.

Assuming  $\beta > \alpha$ , we may in the first case, since the difference of the factors is the constant  $\beta - \alpha$ , put

$$\left. \begin{aligned} x - \alpha &= (\beta - \alpha) \sec^2 \theta \\ x - \beta &= (\beta - \alpha) \tan^2 \theta \end{aligned} \right\},$$

whence

$$dx = 2(\beta - \alpha) \sec^2 \theta \tan \theta d\theta,$$

and

$$\int \frac{dx}{\sqrt{[(x - \alpha)(x - \beta)]}} = 2 \int \sec \theta d\theta = 2 \log (\sec \theta + \tan \theta).$$

Substituting, and omitting the constant  $\log \sqrt{\beta - \alpha}$ ,

$$\int \frac{dx}{\sqrt{[(x - \alpha)(x - \beta)]}} = 2 \log [\sqrt{x - \alpha} + \sqrt{x - \beta}]. \quad (N)$$

In the second case, the sum of the factors being  $\beta - \alpha$ , we may put

$$\left. \begin{aligned} x - \alpha &= (\beta - \alpha) \sin^2 \theta \\ \beta - x &= (\beta - \alpha) \cos^2 \theta \end{aligned} \right\},$$

whence

$$dx = 2(\beta - \alpha) \sin \theta \cos \theta d\theta,$$

and the quantity under the integral sign reduces to  $2d\theta$ . Therefore

$$\int \frac{dx}{\sqrt{[(x - \alpha)(\beta - x)]}} = 2 \sin^{-1} \sqrt{\frac{x - \alpha}{\beta - \alpha}} \dots \dots (O)$$

The same transformations may of course be employed when other factors occur in connection with these radicals

It can be shown that the values given in formulæ (*N*) and (*O*) differ only by constants from the results derived by employing the process given in Art. 56.

### Examples V.

$$1. \int \sqrt[4]{(a-x)} \cdot x \, dx, \quad -\frac{2}{15} (a-x)^{\frac{3}{2}} (3x+2a).$$

$$2. \int \sqrt[4]{(x+a)} \cdot x^3 \, dx, \quad \frac{2}{7} (a+x)^{\frac{7}{2}} - \frac{4a}{5} (a+x)^{\frac{5}{2}} + \frac{2a^2}{3} (a+x)^{\frac{3}{2}}.$$

$$3. \int \frac{x \, dx}{1 + \sqrt[4]{x}}, \quad \frac{2}{3} x^{\frac{3}{2}} - x + 2\sqrt[4]{x} - 2 \log(1 + \sqrt[4]{x}).$$

$$4. \int \frac{x \, dx}{\sqrt[4]{(x+a)}}, \quad \frac{2}{3} (x-2a) \sqrt[4]{(x+a)}.$$

$$5. \int \frac{dx}{\sqrt[4]{x-1}}, \quad 2\sqrt[4]{x} + 2 \log(1 - \sqrt[4]{x}).$$

$$6. \int_{-a}^0 (a+x)^{\frac{1}{2}} x \, dx, \quad \left[ \frac{3y^7}{7} - \frac{3ay^4}{4} \right]_0^{a^{\frac{1}{2}}} = -\frac{9a^{\frac{7}{2}}}{28}.$$

$$7. \int \frac{dx}{x \sqrt{(2ax-a^2)}}, \quad \frac{2}{a} \tan^{-1} \sqrt{\frac{2x-a}{a}}.$$

$$8. \int_0^a (a-x)^{\frac{3}{2}} x^2 \, dx, \quad -\left[ \frac{2y^9}{9} + \frac{4ay^7}{7} - \frac{2a^2y^5}{5} \right]_{\sqrt{a}}^0 = \frac{16a^{\frac{9}{2}}}{315}.$$

$$9. \int \frac{dx}{2x^{\frac{1}{2}} - x^{\frac{1}{3}}}, \quad x^{\frac{1}{2}} + \frac{3x^{\frac{1}{3}}}{4} + \frac{3x^{\frac{1}{6}}}{4} + \frac{3 \log(2x^{\frac{1}{6}} - 1)}{8}.$$

$$10. \int_0^1 (x+1)^{\frac{2}{3}} x \, dx,$$

$$\frac{3y^8}{8} - \frac{3y^5}{5} \Big]_1^{\sqrt[3]{2}} = \frac{3\sqrt[3]{4}}{10} + \frac{9}{40}.$$

$$11. \int \frac{1-3x}{\sqrt{1-x}} \, dx,$$

$$2(1+x)\sqrt{1-x}.$$

$$12. \int \frac{x \, dx}{x - \sqrt{x^2 - a^2}},$$

*Rationalize the denominator.*

$$\frac{x^2 + (x^2 - a^2)^{\frac{3}{2}}}{3a^2}.$$

$$13. \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}},$$

$$\frac{2(x+a)^{\frac{3}{2}} - 2(x+b)^{\frac{3}{2}}}{3(a-b)}$$

$$14. \int \frac{dx}{x\sqrt{x^4+1}},$$

$$\frac{1}{4} \log \frac{\sqrt{x^4+1} - 1}{\sqrt{x^4+1} + 1}.$$

$$15. \int \frac{\sqrt{x^4+1} \, dx}{x},$$

$$\frac{\sqrt{x^4+1}}{2} + \frac{1}{4} \log \frac{\sqrt{x^4+1} - 1}{\sqrt{x^4+1} + 1}.$$

$$16. \int \frac{(x^n+1)(x^n-1)^{\frac{3}{2}}}{x} \, dx,$$

$$\frac{2}{n} \left[ \frac{(x^n-1)^{\frac{5}{2}}}{5} + \frac{(x^n-1)^{\frac{3}{2}}}{3} - (x^n-1)^{\frac{1}{2}} + \tan^{-1} \sqrt{x^n-1} \right].$$

$$17. \int_0^a \frac{x^5 \, dx}{(x^2+a^2)^{\frac{3}{2}}},$$

$$\frac{y^3}{3} - 2a^2 y - \frac{a^4}{y} \Big]_a^{a\sqrt{2}} = \left[ \frac{8}{3} - \frac{11\sqrt{2}}{6} \right] a^3.$$

$$18. \int \frac{\sqrt{x^2-a^2}}{x} \, dx \left[ = \int \frac{x^2-a^2}{x\sqrt{x^2-a^2}} \, dx \right],$$

$$\sqrt{x^2-a^2} - a \sec^{-1} \frac{x}{a}.$$

$$19. \int \frac{x^2 dx}{\sqrt{x^2 + a^2}},$$

$$\left[ = \int \frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}} dx. \text{ See formulas (L) and (K).} \right]$$

$$\frac{1}{2} x \sqrt{x^2 + a^2} - \frac{1}{2} a^2 \log [x + \sqrt{x^2 + a^2}].$$

$$20. \int \frac{\sqrt{a^2 - x^2}}{x} dx, \quad a \log \frac{a - \sqrt{a^2 - x^2}}{x} + \sqrt{a^2 - x^2}.$$

$$21. \int \frac{dx}{x + \sqrt{x^2 + a^2}},$$

$$\frac{x}{2a^2} \sqrt{x^2 + a^2} + \frac{1}{2} \log [x + \sqrt{x^2 + a^2}] - \frac{x^2}{2a^2}.$$

$$22. \int \frac{\sqrt{x^2 + a^2}}{x^2} dx, \quad \text{See Art. 51.}$$

$$\log [\sqrt{x^2 + a^2} + x] - \frac{\sqrt{x^2 + a^2}}{x}.$$

$$23. \int \frac{dx}{\sqrt{x^2 + a^2} - a},$$

$$\log [\sqrt{x^2 + a^2} + x] - \frac{\sqrt{x^2 + a^2}}{x} - \frac{a}{x}.$$

$$24. \int \frac{x dx}{\sqrt{1 + x^4}}. \text{ See Formula (K). } \frac{1}{2} \log [x^2 + \sqrt{1 + x^4}].$$

$$25. \int \sqrt{ax^2 + b} \, dx, \quad [a > 0] \quad \text{Put } \sqrt{ax^2 + b} = z - x\sqrt{a}.$$

$$\frac{b}{2\sqrt{a}} \log [x\sqrt{a} + \sqrt{ax^2 + b}] + \frac{1}{2} x \sqrt{ax^2 + b}.$$

$$26. \int \frac{dx}{(a+x)\sqrt{x^2+b^2}},$$

$$\frac{1}{\sqrt{a^2+b^2}} \log \frac{x + \sqrt{x^2+b^2} + a - \sqrt{a^2+b^2}}{x + \sqrt{x^2+b^2} + a + \sqrt{a^2+b^2}}.$$

$$27. \int \frac{dx}{x^2\sqrt{1+x^2}}, \quad -\frac{\sqrt{1+x^2}}{x}.$$

$$28. \int_{\frac{1}{2}}^1 \frac{dx}{x^2\sqrt{1-x^2}}, \quad -\cot \theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3}.$$

$$29. \int \frac{x^3 dx}{(x^2-a^2)^{\frac{3}{2}}}, \quad \frac{x^2 - 2a^2}{\sqrt{x^2 - a^2}}.$$

$$30. \int \frac{dx}{(p+qx)\sqrt{x^2+1}},$$

$$\frac{1}{\sqrt{p^2+q^2}} \log \tan \frac{1}{2} \left[ \tan^{-1} x + \tan^{-1} \frac{p}{q} \right].$$

$$31. \int \frac{dx}{x^3\sqrt{x^2-1}}, \quad \frac{\sqrt{x^2-1}}{2x^2} + \frac{1}{2} \sec^{-1} x.$$

$$32. \int_0^a \frac{x^2 dx}{(x^2+a^2)^{\frac{3}{2}}}, \quad \log \tan \frac{3\pi}{8} - \frac{\sqrt{2}}{2}.$$

$$33. \int \frac{dx}{x^4 \sqrt{x^2 - 1}}, \quad (2x^2 + 1) \frac{\sqrt{x^2 - 1}}{3x^3}.$$

$$34. \int \frac{dx}{(a-x) \sqrt{a^2 - x^2}}, \quad \frac{1}{a} \sqrt{\frac{a+x}{a-x}}.$$

$$35. \int \frac{dx}{(1+x^2) \sqrt{1-x^2}}, \quad \frac{1}{\sqrt{2}} \tan^{-1} \frac{x \sqrt{2}}{\sqrt{1-x^2}}.$$

$$36. \int_0^a \frac{x^{\frac{5}{2}} dx}{\sqrt{a-x}} \left[ = \int_0^a \frac{x^3 dx}{\sqrt{ax-x^2}} \right]$$

$$\left(\frac{1}{2}a\right)^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin \theta)^3 d\theta = \frac{5\pi a^3}{16}$$

$$37. \int \frac{dx}{x \sqrt{2ax - x^2}}, \quad -\frac{1}{a} \sqrt{\frac{2a-x}{x}}.$$

$$38. \int \frac{dx}{x^3 \sqrt{x^4 - 1}}, \quad \text{Put } x^2 = z = \sec \theta. \quad \frac{\sqrt{x^4 - 1}}{2x^2}.$$

$$39. \int_0^a \sqrt{2ax - x^2} \cdot dx, \quad \frac{a^2 \pi}{4}.$$

$$40. \int_0^a \sqrt{2ax - x^2} \cdot x dx,$$

$$a^3 \int_{-\frac{\pi}{2}}^0 \cos^2 \theta (1 + \sin \theta) d\theta = a^3 \left[ \frac{\pi}{4} - \frac{1}{3} \right].$$

$$41. \int_0^a \sqrt{2ax - x^2} \cdot x^2 dx,$$

$$a^4 \int_{-\frac{\pi}{2}}^0 \cos^2 \theta (1 + \sin \theta)^2 d\theta = a^4 \left[ \frac{5\pi}{16} - \frac{2}{3} \right]$$

$$42. \int \frac{dx}{\sqrt{(2ax + x^2)}},$$

$$\text{by Art. 56,} \quad \log [x + a + \sqrt{(2ax + x^2)}] + C;$$

$$\text{by Art. 58,} \quad 2 \log [\sqrt{x} + \sqrt{(2a + x)}] + C'.$$

$$43. \int \frac{x dx}{\sqrt{(2ax + x^2)}}, \quad \sqrt{(2ax + x^2)} - a \log [x + a + \sqrt{(2ax + x^2)}].$$

$$44. \int \sqrt{\frac{x}{2a - x}} dx \left[ = \int \frac{x dx}{\sqrt{(2ax - x^2)}} \right],$$

$$a \sin^{-1} \frac{x - a}{a} - \sqrt{(2ax - x^2)}.$$

$$45. \int \frac{dx}{\sqrt{(5 + 4x - x^2)}}, \quad \text{by Art. 56,} \quad \sin^{-1} \frac{x - 2}{3} + C;$$

$$\text{by Art. 58,} \quad 2 \sin^{-1} \sqrt{\frac{x + 1}{6}} + C'.$$

$$46. \int_0^a \frac{dx}{\sqrt{(ax - x^2)}}, \quad 2 \sin^{-1} \sqrt{\frac{x}{a}} \Big|_0^a = \pi.$$

$$47. \int \frac{x^2 dx}{\sqrt{(3 + 2x - x^2)}}, \quad 3 \sin^{-1} \frac{x - 1}{2} - \frac{(x + 3) \sqrt{(3 + 2x - x^2)}}{2}.$$

$$48. \int_{-2}^{-\frac{1}{2}} \frac{dx}{\sqrt{(2 - x - x^2)}}, \quad \frac{\pi}{2}.$$

$$49. \int_a^{2a} \frac{dx}{\sqrt{(x^2 - ax)}}, \quad \log (3 + 2\sqrt{2}).$$

50. Find the area included by the rectangular hyperbola

$$y^2 = 2ax + x^2,$$

and the double ordinate of the point for which  $x = 2a$ .

$$a^2[6\sqrt{2} - \log(3 + 2\sqrt{2})].$$

51. Find the area included between the cissoid

$$x(x^2 + y^2) = 2ay^2$$

and the coördinates of the point  $(a, a)$ ; also the whole area between the curve and its asymptote.

$$\left(\frac{3}{4}\pi - 2\right)a^2, \quad \text{and} \quad 3\pi a^2.$$

52. Find the area of the loop of the strophoid

$$x(x^2 + y^2) + a(x^2 - y^2) = 0;$$

also the area between the curve and its asymptote.

$$2a^2\left(1 - \frac{\pi}{4}\right), \quad \text{and} \quad 2a^2\left(1 + \frac{\pi}{4}\right).$$

For the loop put  $y = -x \frac{a+x}{\sqrt{a^2-x^2}}$ , since  $x$  is negative between the limits  $-a$  and  $0$ .

53. Show that the area of the segment of an ellipse between the minor axis and any double ordinate is  $ab \sin^{-1} \frac{x}{a} + xy$ .



## VI.

*Integration by Parts.*

59. Let  $u$  and  $v$  be any two functions of  $x$ ; then since

$$d(uv) = u dv + v du,$$

$$uv = \int u dv + \int v du,$$

whence

$$\int u dv = uv - \int v du \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

By means of this formula, the integration of an expression of the form  $u dv$ , in which  $dv$  is the differential of a known function  $v$ , may be made to depend upon the integration of the expression  $v du$ . For example, if

$$u = \cos^{-1} x \quad \text{and} \quad dv = dx,$$

we have

$$du = -\frac{dx}{\sqrt{1-x^2}};$$

hence, by equation (I),

$$\int \cos^{-1} x \cdot dx = x \cos^{-1} x + \int \frac{x dx}{\sqrt{1-x^2}},$$

in which the new integral is directly integrable; therefore

$$\int \cos^{-1} x \cdot dx = x \cos^{-1} x - \sqrt{1-x^2}.$$

The employment of this formula is called *integration by parts*.

*Geometrical Illustration.*

**60.** The formula for integration by parts may be geometrically illustrated as follows. Assuming rectangular axes, let the curve be constructed in which the abscissa and ordinate of each point are corresponding values of  $v$  and  $u$ , and let this curve cut one of the axes in  $B$ . From any point  $P$  of this curve draw  $PR$  and  $PS$ , perpendicular to the axes. Now the area  $PBOR$  is a value of the indefinite integral  $\int u \, dv$ , and in like

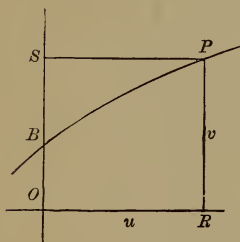


FIG. 2.

manner the area  $PBS$  is a value of  $\int v \, du$ ;

and we have

$$\text{Area } PBOR = \text{Rectangle } PSOR - \text{Area } PBS;$$

therefore

$$\int u \, dv = uv - \int v \, du.$$

*Applications.*

**61.** In general there will be more than one possible method of selecting the factors  $u$  and  $dv$ . The latter of course includes the factor  $dx$ , but it will generally be advisable to include in it any other factors which permit the direct integration of  $dv$ . After selecting the factors, it will be found convenient at once to write the product  $u \cdot v$ , separating the factors by a period; this will serve as a guide in forming the product

$v du$ , which is to be written under the integral sign. Thus, let the given integral be

$$\int x^2 \log x \, dx.$$

Taking  $x^2 dx$  as the value of  $dv$ , since we can integrate this expression directly, we have

$$\begin{aligned} \int x^2 \log x \, dx &= \log x \cdot \frac{1}{3} x^3 - \frac{1}{3} \int x^3 \frac{dx}{x} \\ &= \frac{1}{3} x^3 \log x - \frac{1}{3} \int x^2 dx \\ &= \frac{x^3}{9} (3 \log x - 1). \end{aligned}$$

**62.** The form of the new integral may be such that a second application of the formula is required before a directly integrable form is produced. For example, let the given integral be

$$\int x^2 \cos x \, dx.$$

In this case we take  $\cos x \, dx = dv$ ; so that having  $x^2 = u$ , the new integral will contain a lower power of  $x$ : thus

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx.$$

Making a second application of the formula, we have

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2 \left[ x(-\cos x) + \int \cos x \, dx \right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x. \end{aligned}$$

**63.** The method of integration by parts is sometimes employed with advantage, even when the new integral is no simpler than the given one; for, in the process of successive applications of the formula, the original integral may be reproduced, as in the following example:

$$\begin{aligned} \int \varepsilon^{mx} \sin (nx + \alpha) dx \\ &= \varepsilon^{mx} \cdot \frac{-\cos (nx + \alpha)}{n} + \frac{m}{n} \int \varepsilon^{mx} \cos (nx + \alpha) dx \\ &= -\frac{\varepsilon^{mx} \cos (nx + \alpha)}{n} + \frac{m}{n} \varepsilon^{mx} \frac{\sin (nx + \alpha)}{n} - \frac{m^2}{n^2} \int \varepsilon^{mx} \sin (nx + \alpha) dx, \end{aligned}$$

in which the integral in the second member is identical with the given integral; hence, transposing and dividing,

$$\int \varepsilon^{mx} \sin (nx + \alpha) dx = \frac{\varepsilon^{mx}}{m^2 + n^2} [m \sin (nx + \alpha) - n \cos (nx + \alpha)].$$

**64.** In some cases it is necessary to employ some other mode of transformation, in connection with the method of parts. For example, given the integral

$$\int \sec^3 \theta d\theta;$$

taking  $dv = \sec^2 \theta d\theta$ , we have

$$\int \sec^3 \theta d\theta = \sec \theta \cdot \tan \theta - \int \sec \theta \tan^2 \theta d\theta. \quad \dots \quad (1)$$

If now we apply the method of parts to the new integral, by putting

$$\sec \theta \tan \theta \, d\theta = dv,$$

the original integral will indeed be reproduced in the second member; but it will disappear from the equation, the result being an identity. If, however, in equation (1), we transform the final integral by means of the equation  $\tan^2 \theta = \sec^2 \theta - 1$ , we have

$$\int \sec^3 \theta \, d\theta = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta.$$

Transposing,

$$2 \int \sec^3 \theta \, d\theta = \frac{\sin \theta}{\cos^2 \theta} + \int \frac{d\theta}{\cos \theta};$$

hence, by formula (F), Art. 31,

$$\int \sec^3 \theta \, d\theta = \frac{\sin \theta}{2 \cos^2 \theta} + \frac{1}{2} \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right].$$

### *Formulæ of Reduction.*

**65.** It frequently happens that the new integral introduced by applying the method of parts differs from the given integral only in the values of certain constants. If these constants are expressed algebraically, the formula expressing the first transformation is adapted to the successive transformations of the new integrals introduced, and is called *a formula of reduction*.

For example, applying the method of parts to the integral

$$\int x^n \varepsilon^{ax} dx,$$

we have

$$\int x^n \varepsilon^{ax} dx = x^n \cdot \frac{\varepsilon^{ax}}{a} - \frac{n}{a} \int x^{n-1} \varepsilon^{ax} dx, \quad . \quad . \quad . \quad (I)$$

in which the new integral is of the same form as the given one, the exponent of  $x$  being decreased by unity. Equation (I) is therefore a formula of reduction for this function. Supposing  $n$  to be a positive integer, we shall finally arrive at the integral  $\int \varepsilon^{ax} dx$ , whose value is  $\frac{\varepsilon^{ax}}{a}$ . Thus, by successive application of equation (I) we have

$$\int x^n \varepsilon^{ax} dx = \frac{\varepsilon^{ax}}{a} \left[ x^n - \frac{n}{a} x^{n-1} \dots + (-1)^n \frac{n(n-1) \dots 1}{a^n} \right].$$

$$\text{Reduction of } \int \sin^m \theta d\theta \text{ and } \int \cos^m \theta d\theta.$$

**66.** To obtain a formula of reduction, it is sometimes necessary to make a further transformation of the equation obtained by the method of parts. Thus, for the integral

$$\int \sin^m \theta d\theta,$$

the method of parts gives

$$\int \sin^m \theta d\theta = \sin^{m-1} \theta (-\cos \theta) + (m-1) \int \sin^{m-2} \theta \cos^2 \theta d\theta.$$

Substituting in the latter integral  $1 - \sin^2 \theta$  for  $\cos^2 \theta$ ,

$$\int \sin^m \theta d\theta = -\sin^{m-1} \theta \cos \theta \\ + (m-1) \int \sin^{m-2} \theta d\theta - (m-1) \int \sin^m \theta d\theta;$$

transposing and dividing, we have

$$\int \sin^m \theta d\theta = -\frac{\sin^{m-1} \theta \cos \theta}{m} + \frac{m-1}{m} \int \sin^{m-2} \theta d\theta, \quad \dots \quad (1)$$

a formula of reduction in which the exponent of  $\sin \theta$  is diminished two units. By successive application of this formula, we have, for example:

$$\int \sin^6 \theta d\theta = -\frac{\sin^5 \theta \cos \theta}{6} + \frac{5}{6} \int \sin^4 \theta d\theta \\ = -\frac{\sin^5 \theta \cos \theta}{6} - \frac{5}{6} \frac{\sin^3 \theta \cos \theta}{4} + \frac{5}{6} \frac{3}{4} \int \sin^2 \theta d\theta \\ = -\frac{\sin^5 \theta \cos \theta}{6} - \frac{5 \sin^3 \theta \cos \theta}{6 \cdot 4} - \frac{5 \cdot 3 \sin \theta \cos \theta}{6 \cdot 4 \cdot 2} + \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \theta.$$

**67.** By a process similar to that employed in deriving equation (1), or simply by putting  $\theta = \frac{1}{2}\pi - \theta'$  in that equation, we find

$$\int \cos^m \theta d\theta = \frac{\cos^{m-1} \theta \sin \theta}{m} + \frac{m-1}{m} \int \cos^{m-2} \theta d\theta, \quad \dots \quad (2)$$

a formula of reduction, when  $m$  is positive.

**68.** It should be noticed that, when  $m$  is negative, equation (I) Art. 66 is not a formula of reduction, because the exponent in the new integral is in that case numerically greater than the exponent in the given integral. But, if we now regard the integral in the second member as the given one, the equation is readily converted into a formula of reduction. Thus, putting  $-n$  for the negative exponent  $m - 2$ , whence

$$m = -n + 2,$$

transposing and dividing, equation (I) becomes

$$\int \frac{d\theta}{\sin^n \theta} = -\frac{\cos \theta}{(n-1) \sin^{n-1} \theta} + \frac{n-2}{n-1} \int \frac{d\theta}{\sin^{n-2} \theta} \cdot \cdot \cdot \quad (3)$$

Again, putting  $\theta = \frac{1}{2} \pi - \theta'$  in this equation, we obtain

$$\int \frac{d\theta}{\cos^n \theta} = \frac{\sin \theta}{(n-1) \cos^{n-1} \theta} + \frac{n-2}{n-1} \int \frac{d\theta}{\cos^{n-2} \theta} \cdot \cdot \cdot \quad (4)$$

*Reduction of  $\int \sin^m \theta \cos^n \theta d\theta$ .*

**69.** If we put  $dv = \sin^m \theta \cos \theta d\theta$ , we have

$$\begin{aligned} \int \sin^m \theta \cos^n \theta d\theta &= \frac{\cos^{n-1} \theta \sin^{m+1} \theta}{m+1} \\ &+ \frac{n-1}{m+1} \int \sin^{m+2} \theta \cos^{n-2} \theta d\theta; \cdot \cdot \cdot \quad (I) \end{aligned}$$

but, if in the same integral we put  $dv = \cos^n \theta \sin \theta d\theta$ , we have

$$\begin{aligned} \int \sin^m \theta \cos^n \theta d\theta &= -\frac{\sin^{m-1} \theta \cos^{n+1} \theta}{n+1} \\ &+ \frac{m-1}{n+1} \int \sin^{m-2} \theta \cos^{n+2} \theta d\theta. \cdot \cdot \cdot \quad (2) \end{aligned}$$



When  $m$  and  $n$  are both positive, equation (1) is not a formula of reduction, since in the new integral the exponent of  $\sin \theta$  is increased, while that of  $\cos \theta$  is diminished. We therefore substitute in this integral

$$\sin^{m+2} \theta = \sin^m \theta (1 - \cos^2 \theta),$$

so that the last term of the equation becomes

$$\frac{n-1}{m+1} \int \sin^m \theta \cos^{n-2} \theta d\theta - \frac{n-1}{m+1} \int \sin^m \theta \cos^n \theta d\theta.$$

Hence, by this transformation, the original integral is reproduced, and equation (1) becomes

$$\begin{aligned} \left[ 1 + \frac{n-1}{m+1} \right] \int \sin^m \theta \cos^n \theta d\theta &= \frac{\sin^{m+1} \theta \cos^{n-1} \theta}{m+1} \\ &+ \frac{n-1}{m+1} \int \sin^m \theta \cos^{n-2} \theta d\theta. \end{aligned}$$

Dividing by  $1 + \frac{n-1}{m+1} = \frac{m+n}{m+1}$ , we have

$$\begin{aligned} \int \sin^m \theta \cos^n \theta d\theta &= \frac{\sin^{m+1} \theta \cos^{n-1} \theta}{m+n} \\ &+ \frac{n-1}{m+n} \int \sin^m \theta \cos^{n-2} \theta d\theta, \quad . \quad . \quad (3) \end{aligned}$$

a formula of reduction by which the exponent of  $\cos \theta$  is diminished two units.

By a similar process, from equation (2), or simply by putting  $\theta = \frac{1}{2}\pi - \theta'$  in equation (3), and interchanging  $m$  and  $n$ , we obtain

$$\int \sin^m \theta \cos^n \theta d\theta = -\frac{\sin^{m-1} \theta \cos^{n+1} \theta}{m+n} \\ + \frac{m-1}{m+n} \int \sin^{m-2} \theta \cos^n \theta d\theta, \quad \dots \quad (4)$$

a formula by which the exponent of  $\sin \theta$  is diminished two units.

**70.** When  $n$  is positive and  $m$  negative, equation (1) of the preceding article is itself a formula of reduction, for both exponents are in that case numerically diminished. Putting  $-m$  in place of  $m$ , the equation becomes

$$\int \frac{\cos^n \theta}{\sin^m \theta} d\theta = -\frac{\cos^{n-1} \theta}{(m-1) \sin^{m-1} \theta} - \frac{n-1}{m-1} \int \frac{\cos^{n-2} \theta}{\sin^{m-2} \theta} d\theta. \quad \dots \quad (5)$$

Similarly, when  $m$  is positive and  $n$  negative, equation (2) gives

$$\int \frac{\sin^m \theta}{\cos^n \theta} d\theta = \frac{\sin^{m-1} \theta}{(n-1) \cos^{n-1} \theta} - \frac{m-1}{n-1} \int \frac{\sin^{m-2} \theta}{\cos^{n-2} \theta} d\theta. \quad \dots \quad (6)$$

**71.** When  $m$  and  $n$  are both negative, putting  $-m$  and  $-n$  in place of  $m$  and  $n$ , equation (3) Art. 69 becomes

$$\int \frac{d\theta}{\sin^m \theta \cos^n \theta} = -\frac{1}{(m+n) \sin^{m-1} \theta \cos^{n+1} \theta} \\ + \frac{n+1}{m+n} \int \frac{d\theta}{\sin^m \theta \cos^{n+2} \theta},$$

in which the exponent of  $\cos \theta$  is numerically increased. We

may therefore regard the integral in the second member as the integral to be reduced. Thus, putting  $n$  in place of  $n + 2$ , we derive

$$\int \frac{d\theta}{\sin^m \theta \cos^n \theta} = \frac{1}{(n-1) \sin^{m-1} \theta \cos^{n-1} \theta} + \frac{m+n-2}{n-1} \int \frac{d\theta}{\sin^m \theta \cos^{n-2} \theta} \cdot \cdot \cdot \cdot \cdot \quad (7)$$

Putting  $\theta = \frac{1}{2} \pi - \theta'$ , and interchanging  $m$  and  $n$ , we have

$$\int \frac{d\theta}{\sin^m \theta \cos^n \theta} = - \frac{1}{(m-1) \sin^{m-1} \theta \cos^{n-1} \theta} + \frac{m+n-2}{m-1} \int \frac{d\theta}{\sin^{m-2} \theta \cos^n \theta} \cdot \cdot \cdot \cdot \cdot \quad (8)$$

**72.** The application of the formulæ derived in the preceding articles to definite integrals will be given in the next section. When the value of the indefinite integral is required, it should first be ascertained whether the given integral belongs to one of the directly integrable cases mentioned in Arts. 27 and 28. If it does not, the formulas of reduction must be used, and if  $m$  and  $n$  are integers, we shall finally arrive at a directly integrable form.

As an illustration, let us take the integral

$$\int \sin^2 \theta \cos^4 \theta d\theta.$$

Employing formula (4) Art. 69, by which the exponent of  $\sin \theta$  is diminished, we have

$$\int \sin^2 \theta \cos^4 \theta d\theta = - \frac{\sin \theta \cos^3 \theta}{6} + \frac{1}{6} \int \cos^4 \theta d\theta.$$

The last integral can be reduced by means of formula (2) Art. 67, which, when  $m = 4$ , gives

$$\int \cos^4 \theta d\theta = \frac{\cos^3 \theta \sin \theta}{4} + \frac{3}{4} \int \cos^2 \theta d\theta;$$

therefore

$$\int \sin^2 \theta \cos^4 \theta d\theta = -\frac{\sin \theta \cos^5 \theta}{6} + \frac{\cos^3 \theta \sin \theta}{24} + \frac{\sin \theta \cos \theta}{16} + \frac{\theta}{16},$$

**73.** Again, let the given integral be

$$\int \frac{\cos^6 \theta d\theta}{\sin^3 \theta}.$$

By equation (5), Art. 70, we have

$$\int \frac{\cos^6 \theta d\theta}{\sin^3 \theta} = -\frac{\cos^5 \theta}{2 \sin^2 \theta} - \frac{5}{2} \int \frac{\cos^4 \theta d\theta}{\sin \theta}.$$

We cannot apply the same formula to the new integral, since the denominator  $m - 1$  vanishes; but putting  $n = 4$  and  $m = -1$ , in equation (3) Art. 69, we have

$$\begin{aligned} \int \frac{\cos^4 \theta d\theta}{\sin \theta} &= \frac{\cos^3 \theta}{3} + \int \frac{\cos^2 \theta d\theta}{\sin \theta} \\ &= \frac{\cos^3 \theta}{3} + \int \frac{d\theta}{\sin \theta} - \int \sin \theta d\theta \\ &= \frac{\cos^3 \theta}{3} + \log \tan \frac{1}{2} \theta + \cos \theta. \end{aligned}$$

Hence

$$\int \frac{\cos^6 \theta d\theta}{\sin^3 \theta} = -\frac{\cos^5 \theta}{2 \sin^2 \theta} - \frac{5 \cos^3 \theta}{6} - \frac{5}{2} \log \tan \frac{1}{2} \theta - \frac{5}{2} \cos \theta.$$

## Examples VI.

$$1. \int_0^1 \sin^{-1} x \, dx, \quad x \sin^{-1} x + \sqrt{(1-x^2)} \Big|_0^1 = \frac{\pi}{2} - 1.$$

$$2. \int \sec^{-1} x \, dx, \quad x \sec^{-1} x - \log [x + \sqrt{(x^2 - 1)}]$$

$$3. \int_0^1 \tan^{-1} x \, dx, \quad \frac{\pi}{4} - \frac{\log 2}{2}.$$

$$4. \int x^n \log x \, dx, \quad \frac{x^{n+1}}{n+1} \left[ \log x - \frac{1}{n+1} \right].$$

$$5. \int_0^{\frac{\pi}{2}} \theta \sin \theta \, d\theta, \quad 1.$$

$$6. \int_0^{\frac{\pi}{2}} \theta \cos m\theta \, d\theta, \quad \frac{\pi}{2m} \sin \frac{m\pi}{2} - \frac{2}{m^2} \sin^2 \frac{m\pi}{4}.$$

$$7. \int x \tan^{-1} x \, dx, \quad \frac{1+x^2}{2} \tan^{-1} x - \frac{x}{2}.$$

$$8. \int_0 x^2 e^x \, dx, \quad x^2 e^x - 2x e^x + 2e^x - 2.$$

$$9. \int x \sec^{-1} x \, dx, \quad \frac{1}{2} [x^2 \sec^{-1} x - \sqrt{(x^2 - 1)}].$$

$$10. \int_0^{\frac{\pi}{2}} \theta \sin \left[ \frac{\pi}{4} + \theta \right] d\theta, \quad -\theta \cos \left( \frac{\pi}{4} + \theta \right) + \sin \left( \frac{\pi}{4} + \theta \right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi \sqrt{2}}{4}.$$

$$11. \int x \sec^2 x \, dx, \quad x \tan x + \log \cos x.$$

$$12. \int x \tan^2 x \, dx \left[ = \int x (\sec^2 x - 1) \, dx \right], \quad x \tan x + \log \cos x - \frac{1}{2} x^2.$$

$$13. \int x^2 \sin x \, dx, \quad 2x \sin x + 2 \cos x - x^2 \cos x.$$

$$14. \int_0^1 x \sin^{-1} x \, dx, \quad \frac{1}{2} x^2 \sin^{-1} x \Big|_0^1 - \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta = \frac{\pi}{8}.$$

$$15. \int x^2 \tan^{-1} x \, dx, \quad \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{\log(1+x^2)}{6}.$$

$$16. \int_0^1 x^2 \sin^{-1} x \, dx, \quad \frac{1}{3} x^3 \sin^{-1} x + \frac{2+x^2}{9} \sqrt{1-x^2} \Big|_0^1 = \frac{\pi}{6} - \frac{2}{9}$$

$$17. \int_0^\infty \varepsilon^{-x} \cos x \, dx, \quad \frac{\varepsilon^{-x} (\sin x - \cos x)}{2} \Big|_0^\infty = \frac{1}{2}.$$

$$18. \int \varepsilon^{x \tan \beta} \cos x \, dx, \quad \cos \beta \varepsilon^{x \tan \beta} \sin(\beta + x).$$

$$19. \int \varepsilon^{-x} \sin^2 x \, dx \left[ = \frac{1}{2} \int \varepsilon^{-x} (1 - \cos 2x) \, dx \right],$$

$$\frac{\varepsilon^{-x}}{10} (\cos 2x - 2 \sin 2x - 5).$$

$$20. \int_0^{\frac{\pi}{4}} \varepsilon^\theta \sin \theta \, d\theta, \quad \frac{\varepsilon^\theta}{2} (\sin \theta - \cos \theta) \Big|_0^{\frac{\pi}{4}} = \frac{1}{2}$$

$$21. \int e^x \sin x \cos x \, dx, \quad \frac{e^x}{10} (\sin 2x - 2 \cos 2x).$$

$$22. \int \sin^4 m\theta \, d\theta, \quad \frac{-\sin^3 m\theta \cos m\theta}{4m} + \frac{3\theta}{8} - \frac{3 \sin m\theta \cos m\theta}{8m}.$$

23. Derive a formula of reduction for  $\int (\log x)^n x^m \, dx$ , and deduce from it the value of  $\int (\log x)^3 x^2 \, dx$ .

$$\int (\log x)^n x^m \, dx = (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int (\log x)^{n-1} x^m \, dx.$$

$$\int (\log x)^3 x^2 \, dx = (\log x)^3 \frac{x^3}{3} - (\log x)^2 \frac{x^3}{3} + \frac{2x^3 \log x}{9} - \frac{2x^3}{27}.$$

$$24. \int x \cos^2 x \, dx, \quad \frac{1}{2} x \sin x \cos x - \frac{1}{4} \sin^2 x + \frac{1}{4} x^2.$$

$$25. \int_1 x^2 \sec^{-1} x \, dx, \quad \frac{x^3 \sec^{-1} x}{3} - \frac{x \sqrt{x^2-1}}{6} - \frac{\log [x + \sqrt{x^2-1}]}{6}.$$

26. Derive a formula of reduction for  $\int x^n \sin (x + \alpha) \, dx$ , and deduce from it the value of  $\int x^5 \cos x \, dx$ .

$$\begin{aligned} \int x^n \sin (x + \alpha) \, dx &= -x^n \sin \left[ x + \alpha + \frac{\pi}{2} \right] \\ &\quad + n \int x^{n-1} \sin \left[ x + \alpha + \frac{\pi}{2} \right] dx. \end{aligned}$$

$$\int x^5 \cos x \, dx = (x^5 - 20x^3 + 120x) \sin x + (5x^4 - 60x^2 + 120) \cos x$$

$$27. \int \cos^2 \theta \sin^4 \theta d\theta, \quad \frac{\sin^5 \theta \cos \theta}{6} - \frac{\sin^3 \theta \cos \theta}{24} + \frac{1}{16} [\theta - \sin \theta \cos \theta].$$

$$28. \int_0^{\frac{\pi}{4}} \cos^4 \theta \sin^4 \theta d\theta, \quad \frac{1}{32} \int_0^{\frac{\pi}{2}} \sin^4 \theta' d\theta' = \frac{3\pi}{512}.$$

$$29. \int_0^{\frac{\pi}{4}} \cos^4 \theta d\theta, \quad \frac{\sin \theta \cos^3 \theta}{4} + \frac{3 \sin \theta \cos \theta}{8} + \frac{3\theta}{8} \Big|_0^{\frac{\pi}{4}} = \frac{8 + 3\pi}{32}.$$

$$30. \int_0^{\frac{\pi}{3}} \cos^6 \theta d\theta, \quad \frac{\sin \theta \cos \theta (8 \cos^4 \theta + 10 \cos^2 \theta + 15) + 15\theta}{48} \Big|_0^{\frac{\pi}{3}} = \frac{9\sqrt{3} + 15\pi}{96}.$$

$$31. \int \frac{\cos^4 \theta}{\sin^3 \theta} d\theta, \quad -\frac{\cos^3 \theta}{2 \sin^2 \theta} - \frac{3 \cos \theta}{2} - \frac{3 \log \tan \frac{1}{2} \theta}{2}.$$

$$32. \int \frac{\sin^2 \theta}{\cos^5 \theta} d\theta, \quad \frac{\sin \theta}{4 \cos^4 \theta} - \frac{\sin \theta}{8 \cos^2 \theta} - \frac{1}{8} \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right].$$

$$33. \int \frac{\sin^6 \theta}{\cos^4 \theta} d\theta, \quad \frac{\sin^5 \theta}{3 \cos^3 \theta} - \frac{5 \sin^3 \theta}{3 \cos \theta} + \frac{5}{2} [\theta - \sin \theta \cos \theta].$$

$$34. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^6 \theta}{\sin^2 \theta} d\theta, \quad -\frac{\cos^5 \theta}{\sin \theta} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - 5 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4 \theta d\theta = \frac{48 - 15\pi}{32}.$$

$$35. \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 + \cos \theta)^2}, \quad \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{d\theta'}{\cos^4 \theta'} = \frac{2}{3}.$$

$$36. \int \frac{d\theta}{\sin \theta \cos^4 \theta}, \quad \frac{1}{3 \cos^3 \theta} + \frac{1}{\cos \theta} + \log \tan \frac{\theta}{2}.$$



$$37. \int \frac{d\theta}{\sin \theta \sin^2 2\theta}, \quad \frac{1}{4 \sin^2 \theta \cos \theta} - \frac{3 \cos \theta}{8 \sin^2 \theta} + \frac{3}{8} \log \tan \frac{\theta}{2}.$$

38. Prove that when  $n$  is odd

$$\int \frac{d\theta}{\sin \theta \cos^n \theta} = \frac{\sec^{n-1} \theta}{n-1} + \frac{\sec^{n-3} \theta}{n-3} + \dots + \log \tan \theta;$$

and when  $n$  is even

$$\int \frac{d\theta}{\sin \theta \cos^n \theta} = \frac{\sec^{n-1} \theta}{n-1} + \frac{\sec^{n-3} \theta}{n-3} + \dots + \log \tan \frac{\theta}{2}.$$

$$39. \int \frac{d\theta}{\sin^3 \theta \cos^4 \theta}, \quad -\frac{1}{2 \sin^2 \theta \cos^3 \theta} + \frac{5}{2} \left[ \frac{\sec^3 \theta}{3} + \sec \theta + \log \tan \frac{\theta}{2} \right].$$

$$40. \int \frac{x^2 dx}{\sqrt{x^2-1}}, \quad \text{Put } x = \sec \theta.$$

$$\frac{1}{2}x \sqrt{x^2-1} + \frac{1}{2} \log [x + \sqrt{x^2-1}].$$

$$41. \int (a^2 - x^2)^{\frac{3}{2}} dx, \quad \frac{(5a^2 - 2x^2)x \sqrt{a^2 - x^2}}{8} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a}.$$

$$42. \int_0^a \frac{dx}{(a^2 + x^2)^3}, \quad \frac{1}{a^5} \int_0^{\frac{\pi}{4}} \cos^4 \theta d\theta = \frac{3\pi + 8}{32a^5}.$$

$$43. \int (a^2 + x^2) \sqrt{a^2 - x^2} dx, \quad \frac{5a^4}{8} \sin^{-1} \frac{x}{a} + \frac{x \sqrt{a^2 - x^2} [3a^2 + 2x^2]}{8}$$

$$44. \int \frac{x^2 dx}{(x^2 - a^2)^{\frac{3}{2}}}, \quad -\frac{x}{\sqrt{x^2 - a^2}} + \log [x + \sqrt{x^2 - a^2}]$$

$$45. \int \frac{x^2 dx}{(x^2 + 1)^3}, \quad \frac{x(x^2 - 1)}{8(x^2 + 1)^2} + \frac{\tan^{-1} x}{8}$$

$$46. \int \frac{\cos^3 \theta - \sin^3 \theta}{(\sin \theta + \cos \theta)^2} d\theta \left[ = \int (1 + \sin \theta \cos \theta) \frac{\cos \theta - \sin \theta}{(\sin \theta + \cos \theta)^2} d\theta \right],$$

$$\frac{\sin \theta \cos \theta}{\sin \theta + \cos \theta}.$$

47. Derive a formula for the reduction of  $\int x \sec^n x dx$ ; and referring to Ex. 11, thence show that this is an integrable form when  $n$  is an even integer. Give the result when  $n = 4$ .

$$\int x \sec^n x dx = \frac{x \sec^{n-2} x \tan x}{n-1} - \frac{\sec^{n-2} x}{(n-1)(n-2)}$$

$$+ \frac{n-2}{n-1} \int x \sec^{n-2} x dx.$$

$$\int x \sec^4 x dx = \frac{x \sec^2 x \tan x}{3} - \frac{\sec^2 x}{6} + \frac{2}{3} [x \tan x + \log \cos x].$$

48. Derive a formula of reduction for  $\int x \cos^n x dx$ , and deduce from it the value of  $\int x \cos^3 x dx$ .

$$\int x \cos^n x dx = \frac{x \cos^{n-1} x \sin x}{n} + \frac{\cos^n x}{n^2} + \frac{n-1}{n} \int x \cos^{n-2} x dx.$$

$$\int x \cos^3 x dx = \frac{x \sin x}{3} (\cos^2 x + 2) + \frac{\cos x}{9} (\cos^2 x + 6).$$

49. Find the area between the curve

$$y = \sec^{-1} x.$$

the axis of  $x$ , and the ordinate corresponding to  $x = 2$ .

$$\frac{2\pi}{3} - \log [2 + \sqrt{3}] = 0.77744.$$

50. Find the area between the axis of  $x$ , the curve

$$y = \tan^{-1} x,$$

and the ordinate corresponding to  $x = 1$ .  $\frac{\pi}{4} - \frac{\log 2}{2} = 0.43882.$

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## VII.

### *The Integral and its Limits.*

**74.** Before proceeding to some formulæ of integration involving special values of the limits, it will be convenient to resume the consideration of the integral as defined in the first section.

In Art. 3 we supposed a variable magnitude of which the values depend upon some independent variable  $x$  (but of which the expression as a function of  $x$  is as yet unknown) to vary simultaneously with  $x$ , while  $x$  passes at a uniform rate over a certain range of values. And it is assumed that the rate of the function is then expressed in terms of  $x$  and its assumed rate; or, what is the same thing, that the relative rate of the function as compared with that of  $x$  is known. Denoting this relative rate by  $f(x)$ , a known function of  $x$ , let  $F(x)$  denote the function under consideration; then by the notation introduced

$$F(x) = \int f(x) dx. \quad \dots \quad (I)$$

$F(x)$  may now be defined as the function whose derivative is  $f(x)$ , or whose differential is  $f(x)dx$ ; hence, from this point of view, integration is the search for the indefinite integral, or the inverse of the process of finding the derivative of a function, equation (1) implying nothing more than that

$$d\{F(x)\} = f(x) dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But because this equation was found to be insufficient to fix the values of the function  $F$ , we introduced in Art. 4 the notation of the subscript or lower limit; so that, when we write

$$F(x) = \int_a f(x) dx, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we fully define the value of  $F(x)$  by implying the additional condition that  $F(a) = 0$ . The integral thus modified is sometimes called a *corrected integral*, because the indefiniteness arising from the unknown constant of integration has been removed. It remains "indefinite" in the sense that it is a function of a variable  $x$  to which no special value has been assigned. It is to be noticed that, in many applications, the constant is determined (and the integral thus "corrected") by the condition that some other value (not zero) of  $F(x)$  shall correspond to a given value of  $x$ . These given simultaneous values of  $x$  and  $F(x)$  are called *the initial values*.

**75.** As explained in Art. 4, the value of  $x$  to which corresponds the required value of the magnitude is known as the *final value* of  $x$ , and is used as the upper limit of the integral. Denoting it by  $b$ , we thus assume that, as the independent variable  $x$  passes from  $a$  to  $b$ , the function  $F(x)$  passes *by continuous variation* from its initial to its final value; and, when this is the case, we may write

$$\int_a^b f(x) dx = F(b) - F(a), \quad . \quad . \quad . \quad . \quad (4)$$

in which the function  $F$  is defined, not necessarily by equation (3), which implies  $F(a) = 0$ , but by the more general equation (1), or simply by equation (2).

We may now define the definite integral in equation (4) as *the increment received by a variable whose differential is  $f(x) dx$  while  $x$  passes from the value  $a$  to the value  $b$ ; or, as the magnitude generated at the rate  $f(x) \frac{dx}{dt}$  while  $x$ , with the arbitrary rate  $\frac{dx}{dt}$ , varies from  $a$  to  $b$ .*

The condition mentioned above, that  $F(x)$  shall vary continuously, requires that, starting from some finite initial value  $F(a)$ , it shall not become infinite or imaginary for any value of  $x$  between  $a$  and  $b$ . The function  $F(x)$  cannot become infinite or imaginary unless its derivative  $f(x)$  becomes infinite or imaginary. Hence the condition will be fulfilled if  $f(x)$  remains real and finite for all such values of  $x$ . It is, of course, also assumed that there is no ambiguity about the value of  $f(x)$  corresponding to any of these values of  $x$ . Cases in which such ambiguity might arise will be considered later.

**76.** Since the rate of  $x$  is arbitrary, we may regard  $dx$  in the expression for the integral as constant, and in that case it must be regarded as positive or negative according as  $b$  is greater or less than  $a$ ; in other words,  $dx$  must have the sign of  $b - a$ . When  $f(x)$  does not change sign for any value between  $a$  and  $b$ , we can thus infer the algebraic sign of the definite integral.

Thus, if  $b > a$  and  $f(x)$  is positive for all values between the limits,  $F(x)$  is an increasing function of  $x$ , and  $x$  is itself increasing; therefore the definite integral denotes a positive increment. For example,

$$\int_0^{\pi} \frac{\sin x}{x} dx$$

denotes a positive quantity. Moreover, since in this case  $f(x)$

is less than unity for all values between the limits, the integral is obviously less than the increment received by  $x$ , that is, less than  $\pi$ .

It is evident from these considerations that an interchange of the limits changes the sign of the integral; thus,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx,$$

which agrees also with equation (4). Again, we infer from the same equation that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx,$$

if  $c$  is between the limits  $a$  and  $b$ ; and this is also true when  $c$  is outside of these limits, provided the condition mentioned in the preceding article holds for the entire range of values of  $x$  implied by the several integrals.

**77.** Returning now to the corrected integral, equation (3), we see that it is the same thing as

$$\int_a^x f(x) dx,$$

in which  $x$  stands at once for the upper limit and for the independent variable which, in the generation of the integral, is conceived to vary from the initial to the final value. When there is any danger of confusion between these two meanings of  $x$ , it is well to use some other letter for the independent variable; thus

$$\int_a^x f(z) dz$$

has the same meaning as the expression above; and in general it must be remembered that *an integral is a function of its limits*, and not of the variable which appears under the integral sign unless the same letter serves also to represent one of the limits.

### *Graphic Representation of an Integral.*

**78.** A geometrical illustration was given in Art. 3, in which  $f(x)$  was taken as the ordinate of a curve, and the integral was in consequence represented by an area. We shall now employ another illustration, in which the ordinate  $y$  represents the integral itself, regarded as a function of its upper limit. In other words, we shall consider what is called *the graph of the function* (or graphic representation, employing rectangular coordinates), and shall regard this curve as derived from the expression for the integral, and not from the result of any process of integration. Putting

$$y = \int_a f(x) dx, \quad . . . . . (1)$$

we have

$$\frac{dy}{dx} = f(x) = \tan \phi, \quad . . . . . (2)$$

where  $\phi$  has the same meaning as in Diff. Calc., Art. 26.

Thus for every value of  $x$  we know the inclination  $\phi$  of the curve to the axis of  $x$ . The notation also implies that when  $x = a, y = 0$ . Starting, then, from the position  $(a, 0)$ , if we imagine the point  $(x, y)$  always to move in the proper direction, a direction which changes as  $x$  changes, it will trace out a definite curve, so that the function  $y$  has a definite value for each value of  $x$ .

**79.** As an example, let us take the function

$$y = \int_0^x \frac{\sin x}{x} dx, \quad . . . . . (1)$$

a case in which the indefinite integration cannot be performed. Starting from the origin, the describing point must move in the varying direction defined by

$$\tan \phi = \frac{\sin x}{x}. \quad . . . . . (2)$$

When  $x = 0$ , we have  $\tan \phi = 1$ , or  $\phi = 45^\circ$ ; the curve therefore starts from the origin at this inclination. But, since

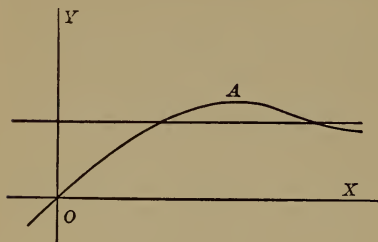


FIG. 3.

equation (2) shows that  $\tan \bar{\phi}$  decreases as  $x$  increases, the curve as we proceed toward the right lies below the tangent at the origin, as in Fig. 3. The ordinate  $y$  continues, however, to increase (compare Art. 76) until, at the point  $A$ ,  $x$  reaches the value  $\pi$  for which

$\phi = 0$ . As  $x$  passes through this value,  $\phi$  changes sign; therefore  $y$  has reached a maximum value. From  $x = \pi$  to  $x = 2\pi$   $\tan \phi$  is negative, hence  $y$  decreases; in this interval  $\sin x$  in equation (2) goes through numerically the same values as before, but the denominator  $x$  is now much larger than before, hence it is plain that  $y$  does not decrease to zero. In like manner it is obvious that it will increase again from  $x = 2\pi$  to  $x = 3\pi$ , and so on, reaching alternate maxima and minima, which continually approach an asymptotic value of  $y$  corresponding to  $x = \infty$ . The form of the curve for positive values of  $x$  is therefore that represented in Fig. 3. For negative values of  $x$  the curve has a similar branch in the third quadrant. It is evident that we can in no case have a finite value of the integral when  $x = \infty$ , unless, as in this case, the quantity under the integral sign approaches zero as a limit when  $x$  increases without limit.

**80.** When the graph or curve of the indefinite integral is drawn, the definite integral between any limits  $c$  and  $d$  is represented by the increment of  $y$  in passing from the abscissa  $c$  to the abscissa  $d$ , and the condition given in Art. 75 requires that the curve shall be continuous between the points corresponding to the limits; in other words, that *these points should belong to the same branch of the curve.*

In the illustration above,  $f(x)$  remains real and finite for *all*



values of  $x$ ; accordingly in this case all values of the limits are possible. But if  $f(x)$  becomes infinite or imaginary for any value of  $x$  between the limits, it will usually be found that, although the indefinite integral  $y = F(x)$  may have a finite value at each limit, the corresponding points will belong to separate branches of the curve, and therefore equation (4) of Art. 75 cannot be used.

For example, if  $f(x) = x^{-2}$ , which is infinite for  $x = 0$ , we have for the indefinite integral

$$y = \int \frac{dx}{x^2} = -\frac{1}{x},$$

and  $y$  is also infinite when  $x = 0$ .

The graph of this integral is an equilateral hyperbola, by means of which we can represent, for example, the definite integral between the limits  $-2$  and  $-1$  as the difference  $BR$  between two values of the ordinate. But we cannot interpret in this case an integral between whose limits the value  $x = 0$  lies (that is, with one negative and one positive limit), because the corresponding points would lie on disconnected branches of the curve.

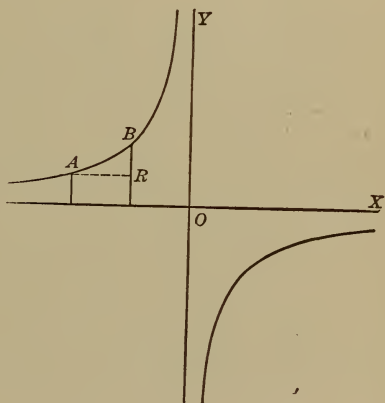


FIG. 4.

It will be noticed that, for the same reason, if we use the notation of the corrected integral we cannot represent the entire curve by a single equation; thus the two branches as drawn in Fig. 4 have the separate equations

$$y = \int_{-\infty}^x \frac{dx}{x^2} \quad \text{and} \quad y = \int_{\infty}^x \frac{dx}{x^2}.$$

the latter expression denoting an essentially negative quantity.

**81.** It may happen, however, that the indefinite integral remains finite for a value of  $x$  which makes  $f(x)$  infinite. In such a case, if the value of  $f(x)$  remains real while  $x$  passes through this value, we shall still have a continuous curve and a continuous variation of the ordinate between limits which

include this value of  $x$ . For example, when  $f(x) = x^{-\frac{1}{2}}$ , which is infinite for  $x = 0$ , and is real for both positive and negative values, we have

$$y = \int \frac{dx}{x^{\frac{1}{2}}} = \frac{3}{2}x^{\frac{3}{2}} + c.$$

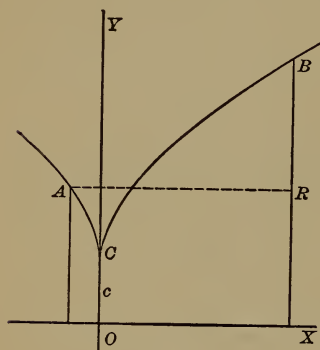


FIG. 5.

The curve which is drawn in Fig. 5 touches the axis of  $y$  and is real on both sides of it, forming a cusp. Thus the ordinate varies continuously while  $x$  passes

through zero, and we can write, for instance,

$$\int_{-1}^8 \frac{dx}{x^{\frac{1}{2}}} = \left[ \frac{3}{2}x^{\frac{3}{2}} \right]_{-1}^8 = 6 - 1\frac{1}{2} = 4\frac{1}{2},$$

in accordance with equation (4), Art. 75, as illustrated in Fig. 5 by the difference of ordinates  $BR$ .

### *Multiple-valued Integrals.*

**82.** When the indefinite integral is a many-valued function, the condition that it shall vary continuously while  $x$  passes from the lower to the upper limit requires the selection of properly corresponding values at the two limits. In illustra-

tion let us construct the graph of the fundamental integral

$$y = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x. \quad \dots \quad (I)$$

In this case  $x$  can only vary between the values  $-1$  and  $+1$ . Proceeding from the origin, the curve, Fig. 6, has the inclination  $45^\circ$ , which is gradually increased to  $90$  when  $x = 1$ , at the point  $A$ , for which the value of  $y$  is  $\frac{1}{2}\pi$ . As  $x$  passes from  $0$  to  $-1$  a similar branch is described in the third quadrant, completing the curve drawn in full line, the ordinate of which is in fact *the primary value* of the indefinite integral  $\sin^{-1} x$  (Diff. Calc., Art. 54). But the complete curve  $y = \sin^{-1} x$ , or  $x = \sin y$ , has another branch continuous with this at the point  $(1, \frac{1}{2}\pi)$ , as represented by the dotted line. In order that the point moving in accordance with the integral expression shall describe this branch, we must suppose that  $x$ , after reaching the value unity at the point  $A$ , begins to decrease, and that, as  $dx$  thus changes sign, the radical  $\sqrt{1-x^2}$ , which then becomes zero, also changes sign, so that the quantity under the integral sign remains positive, and  $y$  goes on increasing. Since the radical cannot change sign, while varying continuously, except when it passes through the value zero (at which time its two values are equal), it is only when  $x = \pm 1$  that  $x$  can return to its previous values without causing  $y$  also to return to the corresponding previous values; that is, without making the generating point return upon the path already described.

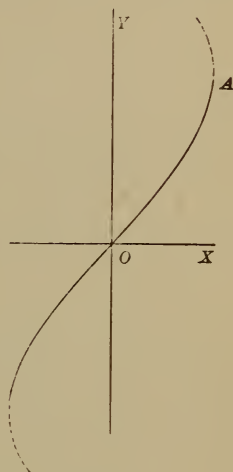


FIG. 6.

We may thus, by the alternate increase and decrease of  $x$  between its extreme values, describe an infinite number of

branches. In this generation of the integral the value of the radical undergoes periodic change of sign but is never ambiguous; for in drawing the curve we assumed that in equation (1) the radical had its positive value when  $x = 0$  and  $y = 0$ . This assumption determines the value of the radical for every other value of  $y$ ; it is, in fact, always equal to  $\cos y$ .

**83.** In the direct application of limits to the integral (1), it is of course sufficient to limit the indefinite integral to its primary value, the radical being positive for all values of  $x$  between the limits; but, when the integral is the result of transformation, care may become necessary in the selection of the values at the limits.

Consider, for example, the integral

$$\int_{-1}^{\sqrt{2}} \frac{dz}{\sqrt{(2-z^2)}},$$

which can be evaluated directly by formula (*j'*) in the form  $\sin^{-1} \frac{z}{\sqrt{2}} \Big]_{-1}^{\sqrt{2}} = \frac{3\pi}{4}$ . But suppose we make the transformation  $z^2 = 1 - x$ , whence  $dz = \frac{-dx}{2z}$ , and  $\sqrt{(2-z^2)} = \sqrt{(1+x)}$ .

It is first to be noticed that, because the value of  $z$  is negative at the lower limit, we should in the value of  $dz$  put  $z = -\sqrt{(1-x)}$ ; thus  $dz = \frac{dx}{2\sqrt{(1-x)}}$ , and the result of transformation is

$$\int_{-1}^{\sqrt{2}} \frac{dz}{\sqrt{(2-z^2)}} = \frac{1}{2} \int_0^{-} \frac{dx}{\sqrt{(1-x^2)}}.$$

The relation  $z^2 = 1 - x$  shows that as  $z$  increases from  $-1$  to  $0$ ,  $x$  increases from  $0$  to  $1$ , and the corrected integral in the second member (which is  $\sin^{-1} x$ ) increases from  $0$  to  $\frac{1}{2}\pi$ . Then as  $z$  further increases from  $0$  to  $\sqrt{2}$ ,  $x$  decreases from  $1$  to  $-1$ ; but the integral continues to increase from  $\frac{1}{2}\pi$  through

values which are not "primary," reaching the final value  $\frac{3}{2}\pi$  at the upper limit, so that we have for the value of the second member of the equation  $\frac{1}{2}(\frac{3}{2}\pi - 0) = \frac{3}{4}\pi$  as before.

**84.** To illustrate another mode in which an integral may acquire multiple-values, let us consider the graph of the fundamental integral

$$y = \int \frac{dx}{1+x^2} = \tan^{-1} x.$$

Proceeding from the origin at the inclination  $45^\circ$ , the curve approaches, as  $x$  increases without limit, an asymptote parallel to the axis of  $x$  at the distance

$\frac{1}{2}\pi$ . The similar branch described for negative values of

$x$  completes the full line in Fig. 7, representing the primary values of  $\tan^{-1} x$ , which

are employed in any direct application of limits to the

integral. But the complete curve  $y = \tan^{-1} x$  consists of

an unlimited number of branches which are repetitions of this curve at successive vertical intervals each equal to  $\pi$ , so that

each asymptote is approached by two branches.

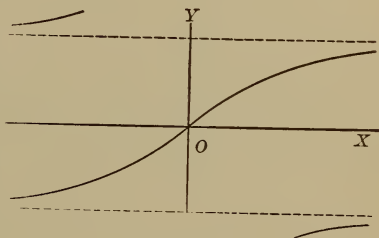


FIG. 7.

Now, when the integral arises from transformation, it may happen that  $x$  passes through infinity, changing sign, and then the ordinate will pass without discontinuity of value from one branch to the next. For example, given the integral

$$\int_0^{\frac{5\pi}{6}} \frac{d\theta}{\cos^2 \theta + 9 \sin^2 \theta} = \int_0^{\frac{5\pi}{6}} \frac{\sec^2 \theta d\theta}{1 + 9 \tan^2 \theta};$$

if we put  $\tan \theta = x$ , this becomes

$$\int_0^{-\frac{\sqrt{3}}{3}} \frac{dx}{1+9x^2} = \frac{1}{3} \tan^{-1} 3x \Big|_0^{-\frac{\sqrt{3}}{3}} = \frac{1}{3} [\tan^{-1}(-\sqrt{3}) - \tan^{-1} 0].$$

As  $\theta$  passes from 0 to  $\frac{1}{2}\pi$ ,  $x$  increases from 0 to  $+\infty$ , and  $\tan^{-1} 3x$  increases from 0 to  $\frac{1}{2}\pi$ . But as  $\theta$  further increases from  $\frac{1}{2}\pi$  to  $\frac{5}{6}\pi$ ,  $x$  changes sign and passes from  $-\infty$  to  $-\frac{1}{3}\sqrt{3}$ . During this interval  $\tan^{-1} 3x$  still further increases from  $\frac{1}{2}\pi$  to  $\frac{2}{3}\pi$ , which is therefore in this case to be taken as the value of  $\tan^{-1} (-\sqrt{3})$ , while we take zero as the value of  $\tan^{-1} 0$ . Hence

$$\int_0^{\frac{5\pi}{6}} \frac{d\theta}{\cos^2 \theta + 9 \sin^2 \theta} = \frac{2\pi}{9}.$$

### *Formulae of Reduction for Definite Integrals.*

**85.** The limits of a definite integral are very often such as to simplify materially the formula of reduction appropriate to it. For example, to reduce

$$\int_0^{\infty} x^n \varepsilon^{-x} dx.$$

we have by the method of parts

$$\int x^n \varepsilon^{-x} dx = -x^n \varepsilon^{-x} + n \int \varepsilon^{-x} x^{n-1} dx.$$

Now, supposing  $n$  positive, the quantity  $x^n \varepsilon^{-x}$  vanishes when  $x=0$ , and also when  $x=\infty$  [See Diff. Calc., Art. 107; Abridged Ed., Art. 91]. Hence, applying the limits 0 and  $\infty$ ,

$$\int_0^{\infty} x^n \varepsilon^{-x} dx = n \int_0^{\infty} x^{n-1} \varepsilon^{-x} dx.$$

By successive application of this formula we have, when  $n$  is an integer,

$$\int_0^{\infty} x^n e^{-x} dx = n(n-1) \cdots \cdots 2 \cdot 1.$$

**86.** From equation (1) Art. 66, supposing  $m > 1$ , we have

$$\int_0^{\frac{\pi}{2}} \sin^m \theta d\theta = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta d\theta.$$

If  $m$  is an integer, we shall, by successive application of this formula, finally arrive at  $\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$  or  $\int_0^{\frac{\pi}{2}} \sin \theta d\theta = 1$ , according as  $m$  is even or odd. Hence

$$\text{if } m \text{ is even, } \int_0^{\frac{\pi}{2}} \sin^m \theta d\theta = \frac{(m-1)(m-3) \cdots \cdots 1}{m(m-2) \cdots \cdots 2} \cdot \frac{\pi}{2}, \dots (P)$$

$$\text{and if } m \text{ is odd, } \int_0^{\frac{\pi}{2}} \sin^m \theta d\theta = \frac{(m-1)(m-3) \cdots \cdots 2}{m(m-2) \cdots \cdots 1} \dots (P')$$

**87.** From equations (3) and (4) Art. 69, we derive

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{n-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^{n-2} \theta d\theta,$$

$$\text{and } \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta d\theta.$$

By successive applications of the first of these formulæ, we produce a series of factors in the numerator decreasing by intervals of two units from  $n - 1$ , and ending with 1 or 2, as in formulas ( $P$ ) and ( $P'$ ). By the second formula we then produce a like series of factors in the numerator beginning with  $m - 1$ . The successive factors in the denominator produced in the process form a single series beginning with  $m + n$  and decreasing by intervals of 2, the final factor being greater by 2 than the sum of the exponents in the final integral. Now this final integral will take one of the four forms

$$\int_0^{\frac{\pi}{2}} d\theta, \quad \int_0^{\frac{\pi}{2}} \sin \theta d\theta, \quad \int_0^{\frac{\pi}{2}} \cos \theta d\theta, \quad \text{or} \quad \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta,$$

according as  $m$  and  $n$  are even or odd. Thus the final factors in the denominator are, in the four cases respectively,

$$2, \quad 3, \quad 3, \quad 4;$$

while the values of the final integrals are respectively

$$\frac{1}{2}\pi, \quad 1, \quad 1, \quad \frac{1}{2}.$$

The result will therefore be the same if we carry the series of factors in the denominator in all cases to 1 or 2, and ignore the final integral except in the first case ( $m$  and  $n$  both even), when its value is  $\frac{1}{2}\pi$ . Therefore we may write, when  $m$  and  $n$  are positive integers,

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)\dots}\alpha, \quad (Q)$$

provided that each series of factors is carried to 2 or 1, and  $\alpha$  is taken equal to unity, except when  $m$  and  $n$  are both even, in which case  $\alpha = \frac{1}{2}\pi$ .

This formula, of course, includes formulæ ( $P$ ) and ( $P'$ ).



*Change of Independent Variable in a Definite Integral.*

**88.** A change of independent variable is frequently made for the purpose of simplifying the limits, rather than of changing the form of the integral. For example, suppose the limits in formula (Q) to be multiples of  $\frac{1}{2}\pi$  by any consecutive integers  $k$  and  $k+1$ . Putting  $\theta = \frac{1}{2}k\pi + \phi$ , we have, according as  $k$  is even or odd,

$$\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \pm \int_0^{\frac{\pi}{2}} \sin^m \phi \cos^n \phi d\phi,$$

or

$$\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \pm \int_0^{\frac{\pi}{2}} \cos^m \phi \sin^n \phi d\phi.$$

But by formula (Q) the integrals in the second members have the same value; hence *the numerical value of the integral of formula (Q) is the same for every quadrant*. The sign to be used is obviously that of  $\sin^m \theta \cos^n \theta$  in the quadrant in question. We can thus readily determine the value of the integral when the limits are any multiples of  $\frac{1}{2}\pi$ . When  $m$  and  $n$  are both even, the sign is always +, and we have to multiply the result of formula (Q) by the number of quadrants. In other cases, we have two positive and two negative results while  $\theta$  describes one revolution.

**89.** When the limits of an integral of the form considered above are 0 and  $\frac{1}{4}\pi$ , we may by introducing the double angle obtain the limits 0 and  $\frac{1}{2}\pi$ . Since, if  $\phi = 2\theta$ ,

$$\sin \theta \cos \theta = \frac{1}{2} \sin \phi, \quad \sin^2 \theta = \frac{1}{2}(1 - \cos \phi), \quad \cos^2 \theta = \frac{1}{2}(1 + \cos \phi),$$

the integral will, *whenever m and n are either both even or both odd*, be thus converted into one or more integrals of the same

form, with integral values of the exponents, and having limits of the desired form.

For example, by this transformation we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^6 \theta \cos^4 \theta d\theta &= \frac{1}{64} \int_0^{\frac{\pi}{2}} \sin^4 \phi (1 - \cos \phi) d\phi \\ &= \frac{1}{64} \left[ \frac{3 \cdot 1 \cdot \pi}{4 \cdot 2 \cdot 2} - \frac{3 \cdot 1 \cdot 1}{5 \cdot 3 \cdot 1} \right] = \frac{1}{64} \left[ \frac{3\pi}{16} - \frac{1}{5} \right].\end{aligned}$$

**90.** A transformation which does not change, or which interchanges, the limits may sometimes reproduce the given integral together with known integrals, and thus lead to its evaluation. When the limits  $a$  and  $b$  of any integral are finite, they will be reversed by the substitution

$$x = a + b - z.$$

Thus

$$\int_a^b f(x) dx = - \int_b^a f(a + b - z) dz = \int_a^b f(a + b - z) dz,$$

or, since it is indifferent whether we write  $z$  or  $x$  for the independent variable in a definite integral,

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx.$$

For example,

$$\int_0^{\pi} \theta \sin^4 \theta d\theta = \int_0^{\pi} (\pi - \theta) \sin^4 (\pi - \theta) d\theta = \int_0^{\pi} (\pi - \theta) \sin^4 \theta d\theta;$$

hence

$$2 \int_0^{\pi} \theta \sin^4 \theta d\theta = \pi \int_0^{\pi} \sin^4 \theta d\theta,$$

and, by formula (P),

$$\int_0^{\pi} \theta \sin^4 \theta d\theta = \pi \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi^2}{16}.$$

91. When the limits are 0 and  $\infty$ , they will be reversed if we assume for the new variable the reciprocal or a multiple of the reciprocal of  $x$ . For example, if in

$$u = \int_0^{\infty} \frac{\log x}{a^2 + x^2} dx$$

we put  $x = \frac{a^2}{y}$ , whence  $dx = -\frac{a^2 dy}{y^2}$ , we find

$$u = \int_0^{\infty} \frac{2 \log a - \log y}{y^2 + a^2} dy = 2 \log a \int_0^{\infty} \frac{dy}{y^2 + a^2} - u;$$

hence

$$u = \log a \int_0^{\infty} \frac{dy}{y^2 + a^2} = \frac{\pi \log a}{2a}.$$

### *The Integral regarded as the Limit of a Sum.*

92. We have seen in Art. 3 that, if the curve  $y = f(x)$  be constructed, the integral  $\int_a^b f(x) dx$  (in which we suppose  $f(x)$  to remain finite and continuous as  $x$  varies from  $a$  to  $b$ ) is represented by the area included by the curve, the axis of  $x$  and the ordinates corresponding to  $x = a$  and  $x = b$ .

Let  $CD$  in Fig. 8 be the curve, and let the base of this area,  $AB$ , whose length is  $b - a$ , be divided into  $n$  equal parts. Denote the length of each part by  $\Delta x$ , so that

$$b = a + n \Delta x,$$

and erect an ordinate as in the figure at each point of division.

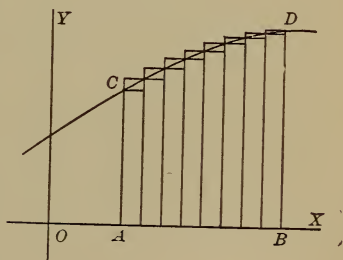


FIG. 8.

The whole area is thus divided into  $n$  parts, as represented by the equation

$$\int_a^b y dx = \int_a^{a+\Delta x} y dx + \int_{a+\Delta x}^{a+2\Delta x} y dx + \dots + \int_{b-\Delta x}^b y dx. \quad (1)$$

Now if we denote the values of  $y$  corresponding to  $a + \Delta x$ ,  $a + 2\Delta x$ ,  $\dots$   $b$  by  $y_1, y_2, \dots y_n$ , the rectangles  $y_1\Delta x, y_2\Delta x, \dots y_n\Delta x$ , which are constructed in the figure, are approximations to the several terms of the second member of equation (1). Hence the sum of these rectangles,

$$\Sigma y \Delta x = y_1 \Delta x + y_2 \Delta x + \dots + y_n \Delta x, \quad (2)$$

is an approximate expression for the integral in equation (1). When, as in Fig. 8,  $y$  increases continuously while  $x$  passes from  $a$  to  $b$ , the sum of the rectangles exceeds the curvilinear area, that is,  $\Sigma y \Delta x$  exceeds the integral.

If we take for the altitude of each rectangle the initial instead of the final value of  $y$  in the interval, we shall obtain in like manner an approximate expression,

$$\Sigma' y \Delta x = y_0 \Delta x + y_1 \Delta x + \dots + y_{n-1} \Delta x, \quad (3)$$

which will, in this case, represent an area less than the curvilinear one, so that  $\Sigma' y \Delta x$  is less than the integral. Thus the integral is intermediate in value to the two expressions in equations (2) and (3), of which the difference is

$$\Sigma y \Delta x - \Sigma' y \Delta x = (y_n - y_0) \Delta x. \quad (4)$$

Therefore the difference between the integral and either of the approximate expressions is less than  $(y_n - y_0) \Delta x$ .

Now when the number of parts  $n$  is indefinitely increased,

so that  $\Delta x$  is decreased without limit, the limit of  $(y_n - y_0) \Delta x$  is zero; it follows that  $\int y dx$  is the limit of  $\sum y \Delta x$  for any range of values of  $x$  for which  $y$  is an increasing function of  $x$ . It can be shown in like manner that the same thing is true while  $y$  is a decreasing function, and therefore in general

$$\int_a^b f(x) dx = \text{the limit of } \sum_{x=a}^{x=b} f(x) \Delta x \quad . \quad . \quad . \quad (5)$$

when  $\Delta x$  decreases without limit.

93. The typical term  $f(x) \Delta x$  is called *the element* of the sum in equation (5). It will be noticed that the sum will have the same limit whether the  $x$  in  $f(x) \Delta x$  be taken as in equation (2) or as in equation (3), or in any other manner, *provided it be some value within the interval to which  $\Delta x$  corresponds*. The expression  $f(x) dx$  is, in like manner, called *the element of the integral* in equation (5).

In many applications of the Integral Calculus, the expression to be integrated is obtained by regarding it as an element. In so doing we are really dealing with the element of the sum; but as we intend to pass to the limit we may, in accordance with the remark made above, ignore the distinction between any values of  $x$  between  $x$  and  $x + \Delta x$ . By equation (5), we pass to the limit by simply writing  $d$  in place of  $\Delta$ , and the integral sign in place of that of summation,\* so that in practice it is customary to form the element of the integral at once by writing  $d$  instead of  $\Delta$ .

---

\* The term *integral*, and the use of the long  $s$ , the initial of the word *sum*, as the sign of integration, have their origin in this connection between the processes of integration and summation.

*Differentiation of an Integral.*

**94.** The integral  $\int_a^x f(x)dx$  is by definition a function of  $x$ , whose derivative, with reference to  $x$ , is  $f(x)$ . Thus, putting

$$U = \int_a^x f(x)dx, \quad \frac{dU}{dx} = f(x).$$

This gives the derivative of an integral with reference to its upper limit. By reversing the limits we have, in like manner.

$$\frac{dU}{da} = -f(a),$$

when the lower limit is regarded as variable.

**95.** Now writing the integral in the form

$$U = \int_a^x u \, dx, \quad \dots \dots \dots (1)$$

if  $u$  is a function of some other quantity,  $\alpha$ , independent of  $x$  and  $a$ ,  $U$  is also a function of  $\alpha$ , and therefore admits of a derivative with reference to  $\alpha$ . From (1) we have

$$\frac{dU}{dx} = u,$$

whence

$$\frac{d}{d\alpha} \frac{dU}{dx} = \frac{du}{d\alpha}.$$

By the principle of differentiation with respect to independent variables [See Diff. Calc., Art. 401; Abridged Ed., Art. 200].

$$\frac{d}{dx} \frac{dU}{d\alpha} = \frac{d}{d\alpha} \frac{dU}{dx}.$$

Therefore

$$\frac{d}{dx} \frac{dU}{d\alpha} = \frac{du}{d\alpha};$$

and by integration

$$\frac{dU}{d\alpha} = \int \frac{du}{d\alpha} dx + C \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, in equation (1),  $U$  is a function of  $x$  and  $\alpha$  which, when  $x = a$ , is equal to zero, independently of the value of  $\alpha$ . In other words, it is a constant with reference to  $\alpha$ , when  $x = a$ ; therefore  $\frac{dU}{d\alpha} = 0$  when  $x = a$ . If, then, we use  $a$  as a lower limit in equation (2), we shall have  $C = 0$ . Therefore

$$\frac{dU}{d\alpha} = \int_a \frac{du}{d\alpha} dx \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting for  $x$  any value  $b$  independent of  $\alpha$ , we have

$$\frac{d}{d\alpha} \int_a^b u dx = \int_a^b \frac{d}{d\alpha} u dx, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which expresses that *an integral may be differentiated with reference to a quantity of which the limits are independent, by differentiating the expression under the integral sign.*

**96.** By means of this theorem, we may derive from an integral whose value is known, the values of certain other integrals. Thus, from the first fundamental integral,

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad . \quad . \quad . \quad . \quad . \quad (I)$$

we derive, by differentiating with reference to  $n$ ,

$$\int x^n \log x \, dx = \frac{(n+1)x^{n+1} \log x - x^{n+1}}{(n+1)^2},$$

the result being the same as that which is obtained by the method of parts.

**97.** The principal application of this method, however, is to definite integrals, when the limits are such as materially to simplify the value of the original integral. Thus, equation (1) of the preceding article gives

$$\int_0^1 x^n \, dx = \frac{1}{n+1},$$

whence, by successive differentiation,

$$\int_0^1 x^n \log x \, dx = -\frac{1}{(n+1)^2},$$

$$\int_0^1 x^n (\log x)^2 \, dx = \frac{1 \cdot 2}{(n+1)^3},$$

.....

$$\int_0^1 x^n (\log x)^r \, dx = (-1)^r \frac{1 \cdot 2 \cdot \dots \cdot r}{(n+1)^{r+1}}.$$

### *Integration under the Integral Sign.*

**98.** Let  $u$  be a function of  $x$  and  $\alpha$ , and let  $a$  and  $\alpha_0$  be constants; then the integral

$$U = \int_{\alpha_0} \left[ \int_a u \, dx \right] d\alpha, \quad \dots \dots \dots (2)$$



is a function of  $x$  and  $\alpha$ , which vanishes when  $\alpha = \alpha_0$ , independently of the value of  $x$ , and when  $x = a$ , independently of the value of  $\alpha$ . From (1)

$$\frac{dU}{d\alpha} = \int_a u \, dx, \quad \text{whence} \quad \frac{d}{dx} \frac{dU}{d\alpha} = u;$$

$$\text{therefore} \quad \frac{d}{d\alpha} \frac{dU}{dx} = u, \quad \text{whence} \quad \frac{dU}{dx} = \int u \, d\alpha + C.$$

Now  $\frac{dU}{dx}$  must vanish when  $\alpha = \alpha_0$ , since this supposition makes  $U$  independent of  $x$ ; therefore, if we use  $\alpha_0$  for a lower limit in the last equation, we must have  $C = 0$ ; therefore

$$\frac{dU}{dx} = \int_{\alpha_0} u \, d\alpha,$$

and since  $U$  vanishes when  $x = a$ ,

$$U = \int_a \left[ \int_{\alpha_0} u \, d\alpha \right] dx. \quad . \quad . \quad . \quad . \quad (2)$$

Comparing the values of  $U$  in equations (1) and (2), we have

$$\int_{\alpha_0} \int_a u \, dx \, d\alpha = \int_a \int_{\alpha_0} u \, d\alpha \, dx.$$

It is evident that we may also write

$$\int_{\alpha_0}^{\alpha_1} \int_a^b u \, dx \, d\alpha = \int_a^b \int_{\alpha_0}^{\alpha_1} u \, d\alpha \, dx, \quad . \quad . \quad . \quad . \quad (3)$$

provided that the limits are all constants.

**99.** By means of this formula, a new integral may be derived from the value of a given integral, provided we can integrate, with reference to the other variable, both the expressions under the integral sign and also the value of the integral. Thus, from

$$\int_0^1 x^n dx = \frac{1}{n+1},$$

by multiplying by  $dn$ , and integrating between the limits  $r$  and  $s$ , we derive

$$\int_0^1 \int_r^s x^n dn dx = \int_r^s \frac{dx}{n+1},$$

whence

$$\int_0^1 \frac{x^s - x^r}{\log x} dx = \log \frac{s+1}{r+1}.$$

**100.** When the derivative of a proposed integral with reference to  $\alpha$  is a known integral, we can sometimes derive its value by integrating the latter with reference to  $\alpha$ . Thus, let

$$u = \int_0^\infty \frac{\varepsilon^{-\alpha x} - \varepsilon^{-\beta x}}{x} dx. \quad \dots \quad (1)$$

In this case

$$\frac{du}{d\alpha} = \int_0^\infty -\varepsilon^{-\alpha x} dx = \left[ \frac{\varepsilon^{-\alpha x}}{\alpha} \right]_0^\infty = -\frac{1}{\alpha};$$

hence, integrating,

$$u = -\log \alpha + C = \log \frac{\beta}{\alpha}, \quad \dots \quad (2)$$

in which  $C$  has been determined by the condition derived from equation (1) that, when  $\alpha = \beta$ ,  $u = 0$ .

*Additional Formulæ of Integration.*

101. The formulæ recapitulated below are useful in evaluating other integrals. (*A*) and (*A'*) are demonstrated in Art. 17; (*B*) and (*C*) in Art. 29; (*D*) and (*E*) in Art. 30; (*F*) in Art. 31; (*G*) and (*G'*) in Art. 35; (*H*) and (*I*) in Art. 50; (*J*) in Art. 51; (*K*) in Art. 52; (*L*) in Art. 53; (*M*) in Art. 55; (*N*) and (*O*) in Art. 58; (*P*) and (*P'*) in Art. 86; and (*Q*) in Art. 87.

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b} \log \frac{x-a}{x-b}. \quad (A)$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}; \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}. \quad (A')$$

$$\int \sin^2 \theta d\theta = \frac{1}{2} (\theta - \sin \theta \cos \theta). \quad (B)$$

$$\int \cos^2 \theta d\theta = \frac{1}{2} (\theta + \sin \theta \cos \theta). \quad (C)$$

$$\int \frac{d\theta}{\sin \theta \cos \theta} = \log \tan \theta. \quad (D)$$

$$\int \frac{d\theta}{\sin \theta} = \log \tan \frac{1}{2} \theta = \log \frac{1 - \cos \theta}{\sin \theta}. \quad (E)$$

$$\int \frac{d\theta}{\cos \theta} = \log \tan \left[ \frac{\pi}{4} + \frac{\theta}{2} \right] = \log \frac{1 + \sin \theta}{\cos \theta} = \log (\sec \theta + \tan \theta). \quad (F)$$

$$\int \frac{d\theta}{a + b \cos \theta} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[ \sqrt{\frac{a-b}{a+b}} \tan \frac{1}{2} \theta \right]. \quad (G)$$

$$\int \frac{d\theta}{a + b \cos \theta} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b+a} + \sqrt{b-a} \tan \frac{1}{2} \theta}{\sqrt{b+a} - \sqrt{b-a} \tan \frac{1}{2} \theta} \quad (G')$$

$$\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \log \frac{\sqrt{x^2 + a^2} - a}{x} \quad (H)$$

$$\int \frac{dx}{x \sqrt{a^2 - x^2}} = \frac{1}{a} \log \frac{a - \sqrt{a^2 - x^2}}{x} \quad (I)$$

$$\int \frac{dx}{(ax^2 + b)^{\frac{3}{2}}} = \frac{x}{b \sqrt{ax^2 + b}} \quad (J)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log [x + \sqrt{x^2 \pm a^2}] \quad (K)$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x \sqrt{x^2 \pm a^2}}{2} \pm \frac{a^2}{2} \log [x + \sqrt{x^2 \pm a^2}] \quad (L)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x \sqrt{a^2 - x^2}}{2} \quad (M)$$

$$\int \frac{dx}{\sqrt{[(x-\alpha)(x-\beta)]}} = 2 \log [\sqrt{x-\alpha} + \sqrt{x-\beta}] \quad (N)$$

$$\int \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}} = 2 \sin^{-1} \sqrt{\frac{x-\alpha}{\beta-\alpha}} \quad (O)$$

$$\int_0^{\pi} \sin^m \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^m \theta d\theta = \frac{(m-1)(m-3) \dots 1}{m(m-2) \dots 2} \cdot \frac{\pi}{2} \quad (P)$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^m \theta \, d\theta = \frac{(m-1)(m-3) \cdots 2}{m(m-2) \cdots 1} \cdot \cdot \cdot \quad (P')$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3) \cdots \times (n-1)(n-3) \cdots}{(m+n)(m+n-2) \cdots} \alpha, \quad (Q)$$

in which  $\alpha = 1$ , unless  $m$  and  $n$  are both even, when  $\alpha = \frac{\pi}{2}$ .

### Examples VII.

$$1. \int_0^{n\pi} \frac{d\theta}{a + b \cos \theta}, \quad [a > b, \text{ and } n \text{ an integer}] \quad \frac{n\pi}{\sqrt{a^2 - b^2}}.$$

$$2. \int_0^{2n\pi \pm \frac{\pi}{2}} \frac{d\theta}{2 + \cos \theta}, \quad \frac{2n\pi \pm \frac{1}{3}\pi}{\sqrt{3}}.$$

$$3. \int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta, \quad \frac{5\pi}{32}.$$

$$4. \int_0^{\pi} \sin^5 \theta \, d\theta, \quad \frac{16}{15}.$$

$$5. \int_{-\frac{\pi}{2}}^{\pi} \cos^7 \theta \, d\theta, \quad \frac{16}{35}.$$

$$6. \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^6 \theta \, d\theta, \quad \frac{3\pi}{512}.$$

$$7. \int_0^{\pi} \sin^3 \theta \cos^4 \theta \, d\theta, \quad \frac{4}{35}.$$

$$8. \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta, \quad \frac{1}{2^n} \int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta.$$

$$9. \int_0^1 \frac{x^{2n} \, dx}{\sqrt{1-x^2}}, \quad \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}.$$

$$10. \int_0^1 \frac{x^{2n+1} \, dx}{\sqrt{1-x^2}}, \quad \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}.$$

$$11. \int_0^a x^3 (a^2 - x^2)^{\frac{5}{2}} \, dx, \quad \frac{2a^9}{63}.$$

$$12. \int_a^{\infty} \frac{(x^2 - a^2)^{\frac{3}{2}} \, dx}{x^6}, \quad \frac{3\pi}{16a}.$$

$$13. \int_0^{\infty} \frac{x^5 \, dx}{(a^2 + x^2)^{\frac{7}{2}}}, \quad \frac{8}{15a}.$$

$$14. \int_0^{\infty} \frac{x^4 \, dx}{(a^2 + x^2)^4}, \quad \frac{\pi}{32a^3}.$$

15. Prove that

$$\int_0^a x^{n-1} (a-x)^{m-1} \, dx = \int_0^a x^{m-1} (a-x)^{n-1} \, dx,$$

and derive a formula of reduction for this integral, supposing  $n > 0$  and  $m > 1$ .

$$\int_0^a x^{n-1} (a-x)^{m-1} \, dx = \frac{m-1}{n} \int_0^a x^n (a-x)^{m-2} \, dx$$

16. Deduce from the result of Ex. 15 the value of the integral when  $m$  is an integer.

$$\int_0^a x^{n-1} (a-x)^{m-1} dx = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (m-1)}{n(n+1) \cdot \dots \cdot (n+m-1)} a^{m+n-1}$$

$$17. \int_{-a}^a (a+x)^5 (a-x)^{\frac{3}{2}} dx. \quad \text{See Ex. 16.} \quad \frac{2^{16} \sqrt{2}}{45 \cdot 45} a^{\frac{15}{2}}$$

$$18. \int_0^{\frac{\pi}{2}} \sin^7 \theta (\cos \theta)^{\frac{5}{2}} d\theta. \quad \text{Put } \sin^2 \theta = x, \text{ and see Ex. 16.}$$

$$\frac{2^8}{5 \cdot 7 \cdot 11 \cdot 19}$$

19. Show by a change of independent variable that

$$\int_0^\infty \frac{x^2 dx}{(a^2 + x^2)^2} = \int_0^\infty \frac{a^2 dx}{(a^2 + x^2)^2},$$

and therefore

$$\int_0^\infty \frac{x^2 dx}{(a^2 + x^2)^2} = \frac{1}{2} \int_0^\infty \frac{dx}{a^2 + x^2} = \frac{\pi}{4a}.$$

$$20. \int_0^\infty \frac{x \log x dx}{(x^2 + a^2)^2}, \quad \frac{\log a}{2a^2}.$$

$$21. \int_0^\infty \frac{\tan^{-1} x \cdot dx}{x^2 + x + 1}, \quad \frac{\pi^2}{6\sqrt{3}}.$$

$$22. \int_0^\infty \tan^{-1} \frac{x}{a} \frac{x dx}{x^4 + a^4}, \quad \frac{\pi^2}{16a^2}.$$

23. Derive a series of integrals by successive differentiation of the definite integral  $\int_0^\infty e^{-ax} dx$ .

$$\int_0^\infty x^n e^{-ax} dx = \frac{1 \cdot 2 \cdot \dots \cdot n}{a^{n+1}}$$

24. Derive from the result of Art. 63 the definite integrals

$$\int_0^{\infty} e^{-mx} \sin nx \, dx = \frac{n}{m^2 + n^2}, \quad \text{and} \quad \int_0^{\infty} e^{-mx} \cos nx \, dx = \frac{m}{m^2 + n^2};$$

and thence derive by differentiation the integrals

$$\int_0^{\infty} x e^{-mx} \sin nx \, dx = \frac{2mn}{(m^2 + n^2)^2}, \quad \text{and} \quad \int_0^{\infty} x e^{-mx} \cos nx \, dx = \frac{m^2 - n^2}{(m^2 + n^2)^2}.$$

25. From the results of Ex. 24 derive

$$\int_0^{\infty} x^2 e^{-mx} \sin nx \, dx = \frac{2n(3m^2 - n^2)}{(m^2 + n^2)^3};$$

$$\int_0^{\infty} x^2 e^{-mx} \cos nx \, dx = \frac{2m(m^2 - 3n^2)}{(m^2 + n^2)^3}.$$

26. From the fundamental formula ( $k'$ ) derive

$$\int_0^{\infty} \frac{dx}{\alpha + \beta x^2} = \frac{\pi}{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}};$$

and thence derive a series of formulas by differentiation with reference to  $\alpha$ .

$$\int_0^{\infty} \frac{dx}{(\alpha + \beta x^2)^n} = \frac{\pi}{2^n \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}}} \cdot \frac{1 \cdot 3 \cdots (2n-3)}{1 \cdot 2 \cdots (n-1)} \cdot \frac{1}{\alpha^{n-\frac{1}{2}}}.$$

27. Derive a series of integrals by differentiating with reference to  $\alpha$  the integral used in Ex. 26.

$$\int_0^{\infty} \frac{x^{2n-2} dx}{(\alpha + \beta x^2)^n} = \frac{\pi}{2^n \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}}} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{1 \cdot 2 \cdot 3 \cdots (n-1)} \cdot \frac{1}{\beta^{n-\frac{1}{2}}}.$$



28. From the integral employed in examples 26 and 27, derive the value of

$$\int_0^{\infty} \frac{x^4 dx}{(\alpha + \beta x^2)^4}.$$

*Differentiate twice with reference to  $\beta$ , and once with reference to  $\alpha$ .*

$$\int_0^{\infty} \frac{x^4 dx}{(\alpha + \beta x^2)^4} = \frac{1 \cdot 3 \cdot 1}{1 \cdot 2 \cdot 3} \cdot \frac{\pi}{16 \alpha^{\frac{3}{2}} \beta^{\frac{5}{2}}}.$$

29. Derive an integral by differentiation, from the result of Ex. II., 67

$$\int_0^{\infty} \frac{dx}{(x^2 + b^2)(x^2 + a^2)^2} = \frac{\pi(2a + b)}{4a^3b(a + b)^2}.$$

30. Derive an integral by integrating  $\int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}.$

$$\int_0^{\infty} \left[ \tan^{-1} \frac{p}{x} - \tan^{-1} \frac{q}{x} \right] \frac{dx}{x} = \frac{\pi}{2} \log \frac{p}{q}.$$

31. Derive a definite integral by integrating

$$\int_0^{\infty} e^{-mx} \sin nx \, dx = \frac{n}{m^2 + n^2}$$

with reference to  $n$ .

$$\int_0^{\infty} \frac{e^{-mx}}{x} (\cos ax - \cos bx) \, dx = \frac{1}{2} \log \frac{m^2 + b^2}{m^2 + a^2}.$$

32. Derive a definite integral from the integral employed in Ex. 31 by integration with reference to  $m$ .

$$\int_0^{\infty} \frac{\sin nx}{x} \left[ e^{-ax} - e^{-bx} \right] dx = \left[ \tan^{-1} \frac{b}{n} - \tan^{-1} \frac{a}{n} \right].$$

33. Derive an integral by integrating with respect to  $m$

$$\int_0^{\infty} e^{-mx} \cos nx \, dx = \frac{m}{m^2 + n^2}.$$

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \cos nx \, dx = \frac{1}{2} \log \frac{b^2 + n^2}{a^2 + n^2}.$$

34. Derive an integral by integrating with respect to  $n$  the integral used in the preceding example.

$$\int_0^{\infty} \frac{e^{-mx}}{x} (\sin ax - \sin bx) \, dx = \tan^{-1} \frac{m(a-b)}{m^2 + ab}.$$

35. Show by means of the result of Ex. 32 that,  $n$  being positive,

$$\int_0^{\infty} \frac{\sin nx}{x} \, dx = \frac{\pi}{2}.$$

36. Derive an integral by integration from the result of Ex. II., 67.

$$\int_0^{\infty} \frac{1}{x} \left[ \tan^{-1} \frac{p}{x} - \tan^{-1} \frac{q}{x} \right] \frac{dx}{x^2 + b^2} = \frac{\pi}{2b^2} \log \frac{p(q+b)}{q(p+b)}.$$

37. Evaluate  $\int_0^{\infty} \log \frac{x^2 + a^2}{x^2 + b^2} \, dx$  by the method of Art. 100.

$$\pi(a-b)$$

38. Evaluate  $\int_0^{\infty} \log \left[ 1 + \frac{a^2}{x^2} \right] \log x \, dx$

$$\pi a (\log a - \frac{1}{2} \pi)$$

## CHAPTER III.

GEOMETRICAL APPLICATIONS—DOUBLE AND TRIPLE  
INTEGRALS.

## VIII.

*Plane Areas.*

**102.** THE first step in making an application of the Integral Calculus is to express the required magnitude in the form of an integral. In the geometrical applications, the magnitude is regarded as generated while some selected independent variable undergoes a given change of value. The independent variable is usually a straight line or an angle, varying between known limits; the required magnitude is either a line regarded as generated by the motion of a point, an area generated by the motion of a line, or a solid generated by the motion of an area. A plane area may be generated by the motion of a straight line, generally of variable length, the method selected depending upon the mode in which the boundaries of the area are defined.

*The Area generated by a Variable Line having a Fixed Direction.*

**103.** The differential of the area generated by the ordinate of a curve, whose equation is given in rectangular coordinates, has been derived in Art. 3. The same method may be employed in the case of any area generated by a straight line whose direction is invariable.

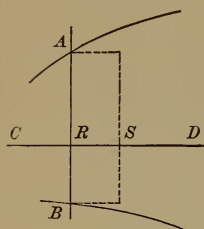


FIG. 9.

Let  $AB$  be the generating line, and let  $R$  be its intersection with a fixed line  $CD$ , to which it is always perpendicular. Suppose  $R$  to move uniformly along  $CD$ , and let  $RS$  be the space described by  $R$  in the interval of time,  $dt$ . Then the value of the differential of the area, at the instant when the generating line passes the position  $AB$ , is the area which would be generated in the time  $dt$ , if the rate of the area were constant. This rate would evidently become constant if the generating line were made constant in length; and therefore the differential is the rectangle, represented in the figure, whose base and altitude are  $AB$  and  $RS$ ; that is, it is *the product of the generating line, and the differential of its motion in a direction perpendicular to its length*.

**104.** In the algebraic expression of this principle, the independent variable is the distance of  $R$  from some fixed origin upon  $CD$ , and the length of  $AB$  is to be expressed in terms of this independent variable.

When the curve or curves defining the length of  $AB$  are given in rectangular coordinates,  $CD$  is generally one of the axes; thus, if the generating line is the ordinate of a curve, the differential is  $y dx$ , as shown in Art. 3. It is often, however, convenient to regard the area as generated by some other line.

For example, given the curve known as the witch, whose equation is

$$y^2 x - 2ay^2 + 4a^2 x = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This curve passes through the origin, is symmetrical to the axis of  $x$ , and has the line  $x = 2a$  for an asymptote, since  $x = 2a$  makes  $y = \pm \infty$ .

Let the area between the curve and its asymptote be re-

quired. We may regard this area as generated by the line  $PQ$  parallel to the axis of  $x$ ,  $y$  being taken as the independent variable. Now

$$PQ = 2a - x,$$

hence the required area is

$$A = \int_{-\infty}^{\infty} (2a - x) dy. \quad \dots (2)$$

From the equation (1) of the curve, we have

$$x = \frac{2ay^2}{y^2 + 4a^2};$$

whence 
$$2a - x = \frac{8a^3}{y^2 + 4a^2},$$

and equation (2) becomes

$$A = 8a^3 \int_{-\infty}^{\infty} \frac{dy}{y^2 + 4a^2} = 4a^2 \tan^{-1} \frac{y}{2a} \Big]_{-\infty}^{\infty} = 4\pi a^2.$$

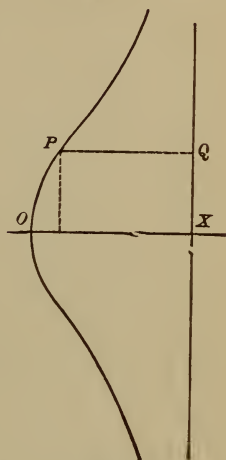


FIG. 10.

### *Oblique Coordinates.*

**105.** When the coordinate axes are oblique, if  $\alpha$  denotes the angle between them, and the ordinate is the generating line, the differential of its motion in a direction perpendicular to its length is evidently  $\sin \alpha \cdot dx$ ; therefore, the expression for the area is

$$A = \sin \alpha \int y dx.$$

As an illustration let the area between a parabola and a chord passing through the focus be required. It is shown in treatises on conic sections, that the value of a focal chord is

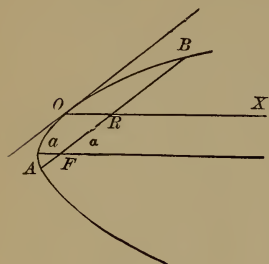


FIG. 11.

$$AB = 4a \operatorname{cosec}^2 \alpha, \quad \dots \quad (1)$$

where  $\alpha$  is the inclination of the chord to the axis of the curve, and  $a$  is the distance from the focus to the vertex. It is also shown that the equation of the curve referred to the diameter which bisects the chord, and the tangent at its extremity which is parallel to the chord is

$$y^2 = 4a \operatorname{cosec}^2 \alpha \cdot x. \quad \dots \quad (2)$$

The required area may be generated by the double ordinate in this equation; and since from (1) the final value of  $y$  is  $\pm 2a \operatorname{cosec}^2 \alpha$ , equation (2) gives for the final value of  $x$

$$OR = a \operatorname{cosec}^2 \alpha.$$

Hence we have

$$A = 2 \sin \alpha \int_0^{a \operatorname{cosec}^2 \alpha} y \, dx,$$

or by equation (2)

$$A = 4 \sin \alpha \int_0^{a \operatorname{cosec}^2 \alpha} \sqrt{x} \, dx = \frac{8a^2 \operatorname{cosec}^3 \alpha}{3}.$$

### *Employment of an Auxiliary Variable.*

106. We have hitherto assumed that, in the expression

$$A = \int y \, dx,$$

$x$  is taken as the independent variable, so that  $dx$  may be assumed constant; and it is usual to take the limits in such a manner that  $dx$  is positive. The resulting value of  $A$  will then have the sign of  $y$ , and will change sign if  $y$  changes sign.

It is frequently desirable, however, as in the illustration given below, to express both  $y$  and  $dx$  in terms of some other variable. When this is done, it is to be noticed that it is not necessary that  $dx$  should retain the same sign throughout the entire integral. The limits may often be so taken that the extremity of the generating ordinate must pass completely around a closed curve, and in that case it is easily seen that the complete integral, which represents the algebraic sum of the areas generated positively and negatively, will be the whole area of the closed curve.

107. As an illustration, let the whole area of the closed curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1,$$

represented in Fig. 12, be required. If in this equation we put

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} = \sin \psi,$$

we shall have

$$\left(\frac{y}{b}\right)^{\frac{1}{3}} = \cos \psi;$$

whence  $x = a \sin^3 \psi$ , and  $y = b \cos^3 \psi$ . . . (1)

Therefore  $\int y dx = 3ab \int \cos^4 \psi \sin^2 \psi d\psi$ .

Now if in this integral we use the limits 0 and  $2\pi$ , the point determined by equation (1) describes the whole curve in the direction  $ABCD A$ . Hence we have for the whole area

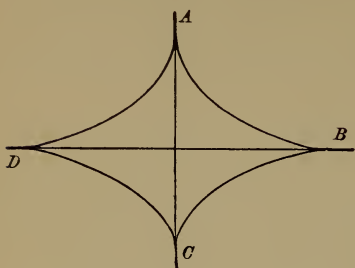


FIG. 12.

$$A = 3ab \int_0^{2\pi} \cos^4 \psi \sin^2 \psi d\psi,$$

and by formula (Q)

$$A = 3ab \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} 2\pi = \frac{3\pi ab}{8}.$$

The areas in this case are all generated with the positive sign, since when  $y$  is negative  $dx$  is also negative. Had the generating point moved about the curve in the opposite direction, the result would have been negative.

### *Area generated by a Rotating Line or Radius Vector.*

**108.** The radius vector of a curve given in polar coordinates is a variable line rotating about a fixed extremity. The angular rate is denoted by  $\frac{d\theta}{dt}$  and may be regarded as constant, but then the rate at which area is generated by the radius vector  $OP$ , Fig. 13, will not be constant, since the length  $OP$  is not constant. The differential of this area is the area which would be generated in the time  $dt$ , if the rate of the area were constant; that is to say, if the radius vector were of constant

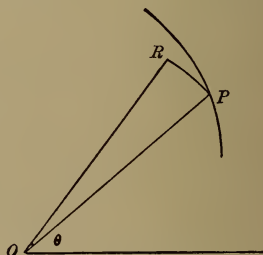


FIG. 13.



length. It is therefore the circular sector  $OPR$  of which the radius is  $r$  and the angle at the centre is  $d\theta$ .

Since  $\text{arc } PR = r d\theta$ ,

$$\text{sector } OPR = \frac{1}{2} r^2 d\theta;$$

therefore the expression for the generated area is

$$A = \frac{1}{2} \int r^2 d\theta \quad \dots \quad (1)$$

109. As an illustration, let us find the area of the right-hand loop of the lemniscata



FIG. 14.

$$r^2 = a^2 \cos 2\theta.$$

The limits to be employed are those values of  $\theta$  which make  $r = 0$ ; that is  $-\frac{\pi}{4}$  and  $\frac{\pi}{4}$ .

Hence the area of the loop is

$$A = \frac{a^2}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta = \frac{a^2}{4} \sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

110. When the radii vectores,  $r_2$  and  $r_1$  corresponding to the same value of  $\theta$  in two curves, have the same sign, the area generated by their difference is the difference of the polar areas generated by  $r_1$  and  $r_2$ . Hence the expression for this area is

$$A = \frac{1}{2} \int (r_2^2 - r_1^2) d\theta \quad \dots \quad (2)$$

III. Let us apply this formula to find the whole area between the cissoid

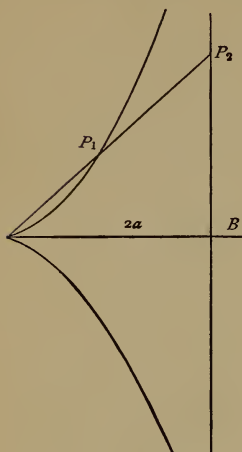


FIG. 15.

$$r_1 = 2a (\sec \theta - \cos \theta),$$

Fig. 15, and its asymptote  $BP_2$ , whose polar equation is

$$r_2 = 2a \sec \theta.$$

One half of the required area is generated by the line  $P_1P_2$ , while  $\theta$  varies from 0 to  $\frac{1}{2}\pi$ . Hence by the formula

$$A = 2a^2 \int_0^{\frac{\pi}{2}} (2 - \cos^2 \theta) d\theta = \frac{3}{2} \pi a^2.$$

Therefore the whole area required is  $3\pi a^2$

### *Transformation of the Polar Formulas.*

112. In the case of curves given in rectangular coordinates, it is sometimes convenient to regard an area as generated by a radius vector, and to use the transformations deduced below in place of the polar formulas.

Put  $y = mx; \dots\dots\dots (1)$

now taking the origin as pole and the initial line as the axis of  $x$ , we have

$$x = r \cos \theta, \quad y = r \sin \theta; \dots\dots (2)$$

therefore  $m = \frac{y}{x} = \tan \theta,$

and  $dm = \sec^2 \theta d\theta. \dots\dots\dots (3)$

From equations (2) and (3),

$$x^2 dm = r^2 d\theta;$$

therefore equation (1) of Art. 108 gives

$$A = \frac{1}{2} \int x^2 dm. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In like manner, equation (2) Art. 110 becomes

$$A = \frac{1}{2} \int (x_2^2 - x_1^2) dm. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

113. As an illustration, let us take the folium

$$x^3 + y^3 - 3axy = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting  $y = mx$ , we have

$$x^3 (1 + m^3) - 3amx^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation gives three roots or values of  $x$ , of which two are always equal zero, and the third is

$$x = \frac{3am}{1 + m^3}; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence

$$y = \frac{3am^2}{1 + m^3}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These are therefore the coordinates of the point  $P$  in Fig. 16. Since the values of  $x$  and  $y$  vanish when  $m = 0$ , and when  $m = \infty$ , the curve has a loop in the first quadrant. To find

the area of this loop we therefore have, by equation (4) of the preceding article,

$$A = \frac{9a^2}{2} \int_0^\infty \frac{m^2 dm}{(1+m^3)^2} = -\frac{3a^2}{2} \left[ \frac{1}{1+m^3} \right]_0^\infty = \frac{3a^2}{2}.$$

114. The area included between this curve and its asymptote may be found by means of equation (5), Art. 112. The equation of a straight line is of the form

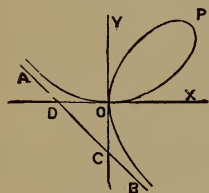


FIG. 16.

$$y = mx + b,$$

and since this line is parallel to  $y = mx$ , the value of  $m$  for the asymptote must be that which makes  $x$  and  $y$  in equations (4) and (5) infinite; that is,  $m = -1$ ; hence the equation of the asymptote is

$$y + x = b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which  $b$  is to be determined. Since when  $m = -1$ , the point  $P$  of the curve approaches indefinitely near to the asymptote, equation (6) must be satisfied by  $P$  when  $m = -1$ . From (4) and (5) we derive

$$y + x = 3a \frac{m^2 + m}{1 + m^3} = \frac{3am}{1 - m + m^2};$$

whence, putting  $m = -1$ , and substituting in equation (6)

$$-a = b,$$

the equation of the asymptote  $AB$ , Fig. 16, is

$$y + x = -a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now, as  $m$  varies from  $-\infty$  to 0, the difference between the radii vectores of the asymptote and curve will generate the areas  $OBC$  and  $ODA$ , hence the sum of these areas is represented by

$$A = \frac{1}{2} \int_{-\infty}^0 (x_2^2 - x_1^2) dm,$$

in which  $x_2$  is taken from the equation of the asymptote (7) and  $x_1$  from that of the curve.

Putting  $y = mx$ , in (7), we have

$$x_2 = -\frac{a}{1+m},$$

and the value of  $x_1$  is given in equation (4). Hence

$$\begin{aligned} A &= \frac{a^2}{2} \int_{-\infty}^0 \left[ \frac{1}{(1+m)^2} - \frac{9m^2}{(1+m^3)^2} \right] dm \\ &= \frac{a^2}{2} \left[ \frac{3}{1+m^3} - \frac{1}{1+m} \right]_{-\infty}^0 \\ &= \frac{a^2}{2} \frac{2+m-m^2}{1+m^3} \Big|_{-\infty}^0 = \frac{a^2}{2} \frac{2-m}{1-m+m^2} \Big|_{-\infty}^0 = a^2. \end{aligned}$$

Adding the triangle  $OCD$ , whose area is  $\frac{1}{2}a^2$ , we have for the whole area required  $\frac{3}{2}a^2$ .

---

\* This reduction is given to show that the integral is not infinite for the value  $m = -1$ , which is between the limits. See Art. 75.

### Examples VIII.

1. Find the area included between the curve

$$a^2y = x^3 + ax^2,$$

and the axis of  $x$ .

$$\frac{a^2}{12}$$

2. Find the whole area of the curve

$$a^4y^2 = x^4(a^2 - x^2).$$

$$\frac{\pi a^3}{4}.$$

3. Find the area of a loop of the curve

$$x^2(a^2 + y^2) = y^2(a^2 - y^2).$$

$$\frac{a^2}{2}(\pi - 2).$$

4. Find the area between the axes and the curve

$$y(x^2 + a^2) = b^2(a - x). \quad b^2 \left[ \frac{\pi}{4} - \frac{\log 2}{2} \right].$$

5. Find the area between the curve

$$x^2y^2 + a^2y^2 - a^2x^2 = 0,$$

and one of its asymptotes.

$$2a^2.$$

6. Find the area between the parabola  $y^2 = 4ax$  and the straight line  $y = x$ .

$$\frac{8a^2}{3}.$$

7. Find the area of the ellipse whose equation is

$$ax^2 + 2bxy + cy^2 = 1.$$

$$\frac{\pi}{4(ac - b^2)}.$$

8. Find the area of the loop of the curve

$$cy^2 = (x - a)(x - b)^2,$$

in which  $c > 0$  and  $b > a$ .

$$\frac{8(b-a)^{\frac{3}{2}}}{15\sqrt{c}}.$$

9. Find the area of the loop of the curve

$$a^3y^2 = x^4(b+x).$$

$$\frac{32b^{\frac{3}{2}}}{105a^{\frac{3}{2}}}.$$

10. Find the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} = 1. \quad \frac{ab}{20}.$$

11. If  $n$  is an integer, prove that the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{n}} + \left(\frac{y}{b}\right)^{\frac{1}{n}} = 1$$

is

$$A = \frac{n(n-1) \cdots 1}{2n(2n-1) \cdots (n+1)} ab.$$

12. If  $n$  is an odd integer, prove that the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{n}} + \left(\frac{y}{b}\right)^{\frac{2}{n}} = 1$$

is

$$A = \frac{[n(n-2) \cdots 1]^2}{2n(2n-2) \cdots 2} \frac{\pi ab}{2}.$$

13. In the case of the curtate cycloid

$$x = a\psi - b \sin \psi,$$

$$y = a - b \cos \psi,$$

find the area between the axis of  $x$  and the arc below this axis.

$$(2a^2 + b^2) \cos^{-1} \frac{a}{b} - 3a \sqrt{b^2 - a^2}.$$

14. If  $b = \frac{1}{2}a\pi$ , show that the area of the loop of the curtate cycloid is

$$\pi a^2 \left[ \frac{\pi^2}{8} - 1 \right].$$

15. Find the area of the segment of the hyperbola

$$x = a \sec \psi,$$

$$y = b \tan \psi,$$

cut off by the double ordinate whose length is  $2b$ .

$$ab \left[ \sqrt{2} - \log \tan \frac{3\pi}{8} \right].$$

16. Find the whole area of the curve

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta.$$

$$\frac{\pi}{2} (a^2 + b^2).$$

17. Find the area of a loop of the curve

$$r^2 = a^2 \cos^2 \theta - b^2 \sin^2 \theta.$$

$$\frac{ab}{2} + \frac{(a^2 - b^2)}{2} \tan^{-1} \frac{a}{b}.$$

18. Find the areas of the large and of each of the small loops of the curve

$$r = a \cos \theta \cos 2\theta;$$



and show that the sum of the loops may be expressed by a single integral.

$$\frac{\pi a^2}{16} + \frac{a^2}{6}, \quad \text{and} \quad \frac{\pi a^2}{32} - \frac{a^2}{12}.$$

19. In the case of the spiral of Archimedes,

$$r = a\theta,$$

find the area generated by the radius vector of the first whorl and that generated by the difference between the radii vectores of the  $n$ th and  $(n + 1)$ th whorl.

$$\frac{4a^2\pi^3}{3} \quad \text{and} \quad 8na^2\pi^3.$$

20. Find the area of a loop of the curve

$$r = a \sin 3\theta. \quad \frac{\pi a^2}{12}.$$

21. Find the area of the cardioid

$$r = 4a \sin^2 \frac{1}{2}\theta. \quad 6\pi a^2.$$

22. Find the area of the loop of the curve

$$r = a \frac{\cos 2\theta}{\cos \theta}. \quad \frac{a^2(4 - \pi)}{2}.$$

23. In the case of the hyperbolic spiral,

$$r\theta = a,$$

show that the area generated by the radius vector is proportional to the difference between its initial and its final value.

24. Find the area of a loop of the curve

$$r = a \cos n \theta, \quad \frac{\pi a^2}{4n}$$

25. Find the area of a loop of the curve

$$r^2 = a^2 \frac{\sin 3\theta}{\cos^2 \theta}, \quad \frac{a^2}{2}$$

26. Find the area of a loop of the curve

$$r^2 \sin \theta = a^2 \cos 2\theta.$$

Notice that  $r$  is real and finite from  $\theta = \frac{5\pi}{4}$  to  $\theta = \frac{7\pi}{4}$ , and that  $\int \frac{d\theta}{\sin \theta}$

is negative in this interval.

$$a^2 \left[ \sqrt{2} - \log (1 + \sqrt{2}) \right].$$

27. Find the area of a loop of the curve

$$(x^2 + y^2)^2 = a^2 xy.$$

Transform to polar coordinates.

$$\frac{a^2}{4}$$

28. In the case of the limaçon

$$r = 2a \cos \theta + b,$$

find the whole area of the curve when  $b > 2a$  and show that the same expression gives the sum of the loops when  $b < 2a$ .

$$(2a^2 + b^2)\pi$$

29. Find separately the areas of the large and small loops of the limaçon when  $b < 2a$ .

$$\text{If } \alpha = \cos^{-1} \left( -\frac{b}{2a} \right),$$

$$\text{large loop} = (2a^2 + b^2) \alpha + \frac{3b}{2} \sqrt{4a^2 - b^2};$$

$$\text{small loop} = (2a^2 + b^2) (\pi - \alpha) - \frac{3b}{2} \sqrt{4a^2 - b^2}.$$

30. Find the area of a loop of the curve

$$r^2 = a^2 \cos n\theta + b^2 \sin n\theta. \qquad \frac{\sqrt{a^4 + b^4}}{n}$$

31. Find the area of the loop of the curve

$$r = a \frac{2 \cos 2\theta - 1}{\cos \theta}, \qquad \left[ 5\sqrt{3} - \frac{8}{3}\pi \right] a^2.$$

32. Show that the sectorial area between the axis of  $x$ , the equilateral hyperbola

$$x^2 - y^2 = 1,$$

and the radius vector making the angle  $\theta$  at the centre is represented by the formula

$$A = \frac{1}{4} \log \frac{1 + \tan \theta}{1 - \tan \theta};$$

and hence show that

$$x = \frac{e^{2A} + e^{-2A}}{2}, \qquad \text{and} \qquad y = \frac{e^{2A} - e^{-2A}}{2}.$$

If  $A$  denotes the corresponding area in the case of the circle

$$x^2 + y^2 = 1,$$

we have

$$x = \cos 2A, \qquad \text{and} \qquad y = \sin 2A.$$

In accordance with the analogy thus presented, the values of  $x$  and  $y$  given above are called the *hyperbolic cosine* and the *hyperbolic sine* of  $2A$ . Thus

$$\frac{e^{2A} + e^{-2A}}{2} = \cosh(2A), \quad \frac{e^{2A} - e^{-2A}}{2} = \sinh(2A).$$

33. Find the area of the loop of the curve

$$x^4 - 3axy^2 + 2ay^3 = 0. \quad \frac{3^6 a^2}{35 \cdot 2^7}.$$

34. Find the area of the loop of the curve

$$x^{2n+1} + y^{2n+1} = (2n+1) ax^n y^n. \quad \frac{2n+1}{2} a^2.$$

35. Find the area between the curve

$$x^{2n+1} + y^{2n+1} = (2n+1) ax^n y^n$$

and its asymptote.

$$\frac{2n+1}{2} a^2.$$

36. Find the area of the loop of the curve

$$y^3 + ax^2 - axy = 0. \quad \frac{a^2}{60}.$$

37. Find the area of a loop of the curve

$$x^4 + y^4 = a^2 xy. \quad \frac{\pi a^2}{8}.$$

38. Trace the curve

$$x = 2a \sin \frac{y}{x},$$

and find the area of one loop.

$$\pi a^2.$$

## IX.

*Volumes of Geometric Solids.*

115. A geometric solid whose volume is required is frequently defined in such a way that the area of the plane section parallel to a fixed plane may be expressed in terms of the perpendicular distance of the section from the fixed plane. When this is the case, the solid is to be regarded as generated by the motion of the plane section, and its differential, when thus considered, is readily expressed.

116. For example, let us consider the solid whose surface is formed by the revolution of the curve  $APB$ , Fig. 17, about the axis  $OX$ . The plane section perpendicular to the axis  $OX$  is a circle; and if  $APB$  be referred to rectangular coordinates, the distance of the section from a parallel plane passing through the origin is  $x$ , while the radius of the circle is  $y$ . Supposing the centre of the section to move uniformly along the axis, the rate at which the volume is generated is not uniform, but its differential is the volume which would be generated while the centre is describing the distance  $dx$ , if the rate were made constant. This differential volume is therefore the cylinder whose altitude is  $dx$ , and the radius of whose base is  $y$ . Hence, if  $V$  denote the volume,

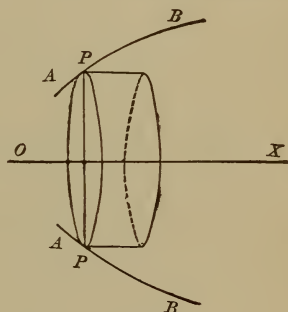


FIG. 17.

$$dV = \pi y^2 dx.$$

117. As an illustration, let it be required to find the volume of the paraboloid, whose height is  $h$ , and the radius of whose base is  $b$ .

The revolving curve is in this case a parabola, whose equation is of the form

$$y^2 = 4ax;$$

and since  $y = b$  when  $x = h$ ,

$$b^2 = 4ah, \quad \text{whence} \quad 4a = \frac{b^2}{h};$$

the equation of the parabola is therefore

$$y^2 = \frac{b^2}{h} x.$$

Hence the volume required is

$$V = \pi \int_0^h y^2 dx = \pi \frac{b^2}{h} \int_0^h x dx = \frac{\pi b^2 h}{2}.$$

118. It can obviously be shown, by the method used in Art. 116, that whatever be the shape of the section parallel to a fixed plane, *the differential of the volume is the product of the area of the generating section and the differential of its motion perpendicular to its plane.*

If the volume is completely enclosed by a surface whose equation is given in the rectangular coordinates  $x, y, z$ , and if we denote the areas of the sections perpendicular to the axes by  $A_x, A_y$ , and  $A_z$ , we may employ either of the formulas

$$V = \int A_x dx, \quad V = \int A_y dy, \quad V = \int A_z dz.$$

The equation of the section perpendicular to the axis of  $x$  is determined by regarding  $x$  as constant in the equation of the surface, and its area  $A_x$  is of course a function of  $x$ .

For example, the equation of the surface of an ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The section perpendicular to the axis of  $x$  is the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - x^2}{a^2},$$

whose semi-axes are  $\frac{b}{a} \sqrt{a^2 - x^2}$  and  $\frac{c}{a} \sqrt{a^2 - x^2}$ .

Since the area of an ellipse is the product of  $\pi$  and its semi-axes,

$$A_x = \frac{\pi bc}{a^2} (a^2 - x^2).$$

The limits for  $x$  are  $\pm a$ , the values between which  $x$  must lie to make the ellipse possible. Hence

$$V = \frac{\pi bc}{a^2} \int_{-a}^a (a^2 - x^2) dx = \frac{4\pi abc}{3}.$$

**119.** The area  $A_x$  can frequently be determined by the conditions of the problem without finding the equation of the surface. For example, let it be required to find the volume of the solid generated by so moving an ellipse with constant major axis, that its center shall describe the major axis of a fixed ellipse, to whose plane it is perpendicular, while the extremities of its minor axis describe the fixed ellipse. Let the equation of the fixed ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let  $c$  be the major semi-axis of the moving ellipse. The minor semi-axis of this ellipse is  $y$ . Since the area of an ellipse is equal to  $\pi$  multiplied by the product of its semi-axes, we have

$$A_x = \pi cy = \frac{\pi cb}{a} \sqrt{(a^2 - x^2)}.$$

Therefore 
$$V = \frac{\pi bc}{a} \int_{-a}^a \sqrt{(a^2 - x^2)} dx;$$

hence, see formula (M),

$$V = \frac{\pi^2 abc}{2}.$$

*The Solid of Revolution regarded as Generated by a Cylindrical Surface.*

120. A solid of revolution may be generated in another manner, which is sometimes more convenient than the employment of a circular section, as in Art. 116. For example, let the cissoid  $POR$ , Fig. 18, whose equation is

$$y^2(2a - x) = x^3,$$

revolve about its asymptote  $AB$ . The line  $PR$ , parallel to  $AB$  and terminated by the curve, describes a cylindrical surface. If we conceive the radius of this cylinder to

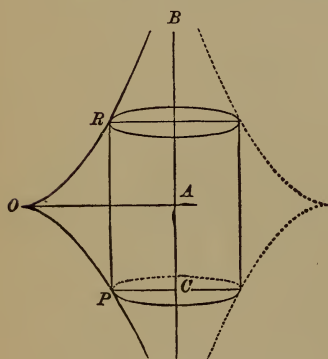


FIG. 18.

pass from the value  $OA = 2a$  to zero, the cylindrical surface will evidently generate the solid of revolution. Now every



point of this cylindrical surface moves with a rate equal to that of the radius; therefore the differential of the solid is the product of the cylindrical surface, and the differential of the radius. The radius and altitude in this case are

$$PC = 2a - x, \quad \text{and} \quad PR = 2y,$$

therefore 
$$V = 4\pi \int_0^{2a} (2ax - x^2)^{\frac{1}{2}} x \, dx.$$

Putting 
$$x - a = a \sin \theta,$$

$$V = 4\pi a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 \theta + \cos^2 \theta \sin \theta) \, d\theta = 2\pi^2 a^3.$$

### Examples IX.

1. Find the volume of the spheroid produced by the revolution of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

about the axis of  $x$ .

$$\frac{4\pi ab^3}{3}.$$

2. Find the volume of a right cone whose altitude is  $a$ , and the radius of whose base is  $b$ .

$$\frac{\pi ab^2}{3}.$$

3. Find the volume of the solid produced by the revolution about the axis of  $x$  of the area between this axis, the cissoid

$$y^3 (2a - x) = x^3,$$

and the ordinate of the point  $(a, a)$ .

$$8a^3\pi (\log 2 - \frac{2}{3}).$$

4. Find the volume generated by the revolution of the witch,

$$y^2x - 2ay^2 + 4a^2x = 0,$$

about its asymptote.

See Art. 104.

$$4\pi^2a^3.$$

5. The equilateral hyperbola

$$x^2 - y^2 = a^2$$

revolves about the axis of  $x$ : show that the volume cut off by a plane cutting the axis of  $x$  perpendicularly at a distance  $a$  from the vertex is equal to a sphere whose radius is  $a$ .

6. An anchor ring is formed by the revolution of a circle whose radius is  $b$  about a straight line in its plane at a distance  $a$  from its centre: find its volume.

$$2\pi^2ab^2.$$

7. Express the volume of a segment of a sphere in terms of the altitude  $h$  and the radii  $a_1$  and  $a_2$  of the bases.

$$\frac{\pi h}{6} (h^2 + 3a_1^2 + 3a_2^2).$$

8. Find the volume generated by the revolution of the cycloid,

$$x = a(\psi - \sin \psi),$$

$$y = a(1 - \cos \psi),$$

about its base.

$$5\pi^2a^3.$$

9. The area included between the cycloid and tangents at the cusps and at the vertex revolves about the latter; find the volume generated.

$$\pi^2a^3.$$

10. Find the volume generated by the revolution of the part of the curve

$$y = e^x,$$

which is on the left of the origin, about the axis of  $x$ .

$$\frac{\pi}{2}.$$

11. The axes of two equal right circular cylinders, whose common radius is  $a$ , intersect at the angle  $\alpha$ ; find the volume common to the cylinders.

*The section parallel to the axes is a rhombus.*

$$\frac{16a^3}{3 \sin \alpha}.$$

12. Find the volume generated by the revolution of one branch of the sinusoid,

$$y = b \sin \frac{x}{a},$$

about the axis of  $x$ .

$$\frac{\pi^2 a b^2}{2}.$$

13. Find the volume enclosed by the surface generated by the revolution of an arc of a parabola about a chord, whose length is  $2c$ , perpendicular to the axis, and at a distance  $b$  from the vertex.

$$\frac{16\pi b^2 c}{15}.$$

14. Find the volume generated by the revolution of the tractrix, whose differential equation is

$$\frac{dy}{dx} = \pm \frac{y}{\sqrt{a^2 - y^2}},$$

about the axis of  $x$ .

*Express  $\pi y^2 dx$  in terms of  $y$ .*

$$\frac{2\pi a^3}{3}.$$

15. Find the volume generated by the curve

$$xy^2 = 4a^2 (2a - x)$$

revolving about its asymptote.

$$4\pi^2 a^3.$$

16. Express the volume of a frustum of a cone in terms of its height  $h$ , and the radii  $a_1$  and  $a_2$  of its bases.

$$\frac{\pi h}{3} (a_1^3 + a_1 a_2 + a_2^3).$$

17. Find the volume of a barrel whose height is  $2h$ , and diameter  $2b$ , the longitudinal section through the centre being a segment of an ellipse whose foci are in the ends of the barrel.

$$2\pi b^2 h \frac{2h^2 + 3b^2}{3(b^2 + h^2)}.$$

18. Find the volume generated by the superior and by the inferior branch of the conchoid each revolving about the directrix; the equation, when the axis of  $y$  is the directrix, being

$$x^2 y^2 = (a + x)^2 (b^2 - x^2).$$

$$\pi a b^3 \pm \frac{4\pi b^3}{3}.$$

19. The area included between a quadrant of the ellipse

$$x = a \cos \phi, \quad y = b \sin \phi,$$

and the tangents at its extremities revolves about the tangent at the extremity of the minor axis; find the volume generated.

$$\frac{\pi a b^2 (10 - 3\pi)}{6}.$$

20. An ellipse revolves about the tangent at the extremity of its major axis; express the entire volume in the form of an integral, whose limits are 0 and  $2\pi$ , and find its value.

$$2\pi^2 a^2 b.$$

21. Find the volume generated by the revolution of a circular arc whose radius is  $a$  about its chord whose length is  $2c$ .

$$\frac{2\pi c (3a^2 - c^2)}{3} - 2\pi a^2 \sqrt{a^2 - c^2} \sin^{-1} \frac{c}{a}.$$

22. A straight line of fixed length  $2c$  moves with its extremities in two fixed perpendicular straight lines not in the same plane, and at a distance  $2b$ . Prove that every point in the moving line describes an ellipse in a plane parallel to both the fixed lines, and find the volume enclosed by the generated surface.

$$\frac{4\pi (c^3 - b^3)b}{3}.$$

## X.

*Double Integrals.*

121. Let us consider the expression

$$\int_{y_1}^{y_2} \phi(x, y) dy, \quad . . . . . (1)$$

in which the limits  $y_1$  and  $y_2$  may be any functions of  $x$ . In the integration,  $x$  is to be treated like any other quantity independent of  $y$  which may be involved in the expression  $\phi(x, y)$ ; in other words, as a constant with respect to the variable  $y$ . Thus the *indefinite* integral will contain both  $x$  and  $y$ ; but since the *definite* integral is a function not of the independent variable but of the limits, the expression (1) is independent of  $y$ , but is generally a function of  $x$ . We may therefore denote it by  $f(x)$ , and write it in place of  $f(x)$  in the expression of an integral in which  $x$  is the independent variable. Thus, putting

$$\int_{y_1}^{y_2} \phi(x, y) dy = f(x), \quad . . . . . (2)$$

$$\int_a^b f(x) dx = \int_a^b \int_{y_1}^{y_2} \phi(x, y) dy dx. \quad . . . . . (3)$$

In the last expression, which is called a *double integral*,  $x$  and  $y$  are two independent variables. It is to be noticed that the limits of the  $y$ -integration, which is to be performed first, may be functions of the other variable  $x$ , but the limits of the final integration must be constants, that is, independent both of  $x$  and of  $y$ .

122. In accordance with Art. 92, the expression (1) is the limit of  $\sum_{y_1}^{y_2} \phi(x, y) \Delta y$ , when  $\Delta y$  is diminished without limit, being assumed that  $\phi(x, y)$  is finite and continuous while  $y$  passes from  $y_1$  to  $y_2$ . Further, assuming this to be the case

for all values of  $x$  from  $a$  to  $b$  (so that  $f(x)$  in equation (3) is finite and continuous for all values concerned in the  $x$ -integration), the double integral is the limit of  $\sum_a^b f(x) \Delta x$ , when  $\Delta x$  is diminished without limit. It readily follows that the double integral is the limit of

$$\sum_a^b \sum_{y_1}^{y_2} \phi(x, y) \Delta y \Delta x,$$

where both  $\Delta y$  and  $\Delta x$  are diminished without limit.\*

The typical term  $\phi(x, y) \Delta y \Delta x$  is called *the element of the sum*, and in like manner  $\phi(x, y) dy dx$  is *the element of the double integral*. As mentioned in Art. 93, in forming the expression for the element, no distinction need be drawn between any values of  $x$  and  $y$ , between  $x$  and  $x + \Delta x$ ,  $y$  and  $y + \Delta y$ , respectively, because these distinctions disappear at the limit.

### *Limits of the Double Integral.*

**123.** In discussing the limits of the double integral (3), Art. 121, it is convenient to consider first the simpler expression

$$\int_a^b \int_{y_1}^{y_2} dy dx. \quad \dots \quad (1)$$

Performing the  $y$ -integration, this reduces to the simple integral

$$\int_a^b (y_2 - y_1) dx. \quad \dots \quad (2)$$

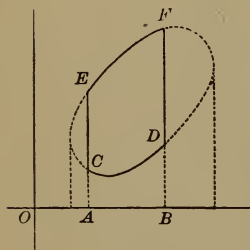


FIG. 19.

\* Denoting the difference between  $f(x)$  and  $\sum_{y_1}^{y_2} \phi(x, y) \Delta y$ , of which it is the limit, by  $e$ ,  $e$  is a quantity which vanishes with  $\Delta y$ . Then the difference between  $\sum_a^b f(x) \Delta x$  and  $\sum_a^b \sum_{y_1}^{y_2} \phi(x, y) \Delta y \Delta x$  is  $\sum_a^b e \Delta x$ , a quantity which vanishes with  $\Delta y$  if  $a$  and  $b$  are finite. Hence, in this case, the double integral which is the limit of the first of these sums is also the limit of the second. But the conclusion does not follow when the limits are infinite; in fact, the double integral is not then always independent of the mode in which the limits become infinite.

In Fig. 19, using rectangular coordinates, let  $OA = a$ ,  $OB = b$ , and let  $CD, EF$  be the curves  $y = y_1, y = y_2$ ; then (2) is, by Art. 103, the expression for the area  $CDFE$ . Therefore the double integral (1) is represented by *the area enclosed by the curves  $y = y_1, y = y_2$  and the straight lines  $x = a, x = b$ .*

**124.** In order that the double integral may represent the area enclosed by a single curve (like the dotted line of Fig. 19) of which the equation is known, the limits  $y_1$  and  $y_2$  must be two values of  $y$  corresponding to the same value of  $x$ , and the limits for  $x$  must be those values for which  $y_1$  and  $y_2$  are equal. Between the limiting values of  $x$  the values of  $y$  are real, and beyond them the values of  $y$  become imaginary. For example, suppose the curve to be the ellipse

$$2x^2 - 2xy + y^2 - 4x - 2y + 6 = 0;$$

solving for  $y$ ,

$$y = x + 1 \pm \sqrt{(-x^2 + 6x - 5)} = x + 1 \pm \sqrt{(x - 1)(5 - x)};$$

whence

$$y_2 - y_1 = 2\sqrt{(x - 1)(5 - x)}.$$

This expression is real for all values of  $x$  between 1 and 5; hence the entire area is

$$A = \int_1^5 \int_{y_1}^{y_2} dy dx = 2 \int_1^5 \sqrt{(x - 1)(5 - x)} dx^* = 4\pi.$$

It is evident that we might equally well have used the expression

$$A = \int_p^q \int_{x_1}^{x_2} dx dy = \int_p^q (x_2 - x_1) dy,$$

---

\* It is useful to notice that a definite integral of this form, by a familiar property of the circle, represents the area of the semicircle whose diameter is the difference between the limits.

in which  $x_1$  and  $x_2$  are obtained as functions of  $y$ , by solving the equation of the curve for  $x$ , thus

$$x = \frac{1}{2}(y + 2) \pm \frac{1}{2}\sqrt{(-y + 8y - 8)};$$

and the limiting values of  $y$  are those obtained by putting the radical equal to zero, namely,  $p = 4 - 2\sqrt{2}$ ,  $q = 4 + 2\sqrt{2}$ . Accordingly the same area is expressed by

$$A = \int_{4-2\sqrt{2}}^{4+2\sqrt{2}} \sqrt{(-y^2 + 8y - 8)},$$

whence  $A = 4\pi$  as before.

**125.** It appears therefore that no distinction is to be drawn between the expressions

$$\iint dy dx \quad \text{and} \quad \iint dx dy.$$

We may in fact regard either one as representing any area whatever, the value becoming definite only when we assign a defined closed *contour* or boundary line; just as  $\int dx$  may represent a line of any length measured along the axis of  $x$ , and is definite only when we assign the limits which determine its two extremities. Thus the contour bears the same relation to the double integral that the pair of limits does to the simple integral.

When the boundary of a given area is made up of lines having different equations, it is not generally possible to express the area by means of a single double integral. This was possible, it is true, in the case of the area enclosed by the full line in Fig. 19, because the equations of two of the bounding lines,  $x = a$  and  $x = b$ , contained only one variable, and the integration with respect to this was made the final one in expression (1). But, if integration with respect to  $x$  had been performed first, it would have been necessary to break the area



up into several parts, of which it is the algebraic sum. In the expressions for these parts, the limits for  $x$  would be taken from the equations of the different lines, and the final limits would be the ordinates, which it would be necessary to find (instead of the known abscissas), of the intersections  $C$ ,  $D$ ,  $E$ , and  $F$ .

**126.** Returning now to the general double integral

$$\int_a^b \int_{y_1}^{y_2} \phi(x, y) dy dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and employing rectangular coordinates, let the area  $ASBR$  in the plane of  $xy$ , Fig. 20, be that which is defined by the limits of the given integral, in other words, the area represented by

$$\int_a^b \int_{y_1}^{y_2} dy dx. \quad . \quad . \quad . \quad (2)$$

In evaluating the double integral (1) with the given limits, the integration of the element  $\phi(x, y) dy dx$  is said to *extend over the area*  $ASBR$  represented by expression (2).

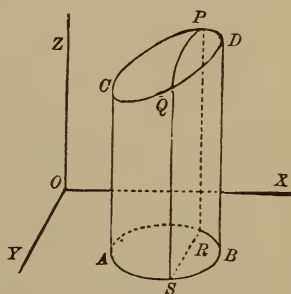


FIG. 20.

Now let us construct the surface whose equation is

$$z = \phi(x, y). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The  $z$ -coordinates of all points whose projections on the plane of  $xy$  are on the curve  $ASBR$  lie in a cylindrical surface, which in Fig. 20 is supposed to cut the surface (3) in the curve  $CQDP$ . It is assumed that  $\phi(x, y)$ , or  $z$ , remains finite and continuous while the independent variables  $x$  and  $y$  vary in any way, provided the point  $(x, y)$  remains within the area  $ASBR$ . This assumption is clearly identical with that made in Art. 122. A definite solid will then be enclosed between the surface (3),

the plane of  $xy$ , and the cylindrical surface. We have now to show that this solid will represent the double integral (1).

Let  $SRPQ$  be a section of this solid by a plane parallel to that of  $yz$ , so that in it  $x$  has a constant value. The ordinates of the points  $R$  and  $S$  are the values of  $y_1$  and  $y_2$  corresponding to that particular value of  $x$ . The section  $SRPQ$  may be regarded as generated by the line  $z$ , while  $y$  varies from  $y_1$  to  $y_2$ ; hence, denoting its area by  $A_x$ , as in Art. 118,

$$A_x = \int_{y_1}^{y_2} z \, dy.$$

Therefore the volume is

$$V = \int_a^b A_x \, dx = \int_a^b \int_{y_1}^{y_2} z \, dy \, dx$$

which is identical with expression (1).

### *Change of the Order of Integration.*

**127.** It is obvious that the order of integration may be reversed as in Art. 124, provided that the integration extends over the same area. Considering the corresponding process of double summation, we may be said, in the first case, to sum up those of the elements  $z \, dy \, dx$  which have a common value of  $x$  into the sum  $A_x \, dx$ , which then constitutes the element of the second summation; while in the second case we first sum up those of the original elements which have a common value of  $y$  into the element  $A_y \, dy$ , which is afterward summed for all values of  $y$  within the given limits.

Since, in expression (1), Art. 126,  $dx$  is a constant with respect to the  $y$ -integration, it may be removed to the left of the corresponding integral sign. In the resulting expression,

$$\int_a^b dx \int_{y_1}^{y_2} \phi(x, y) \, dy,$$

it is to be understood that, because the  $y$ -integral which is a function of  $x$  follows the sign of  $x$ -integration, it is "under" it, or subject to it. This notation has the advantage of making it perfectly clear to which variable each integral sign corresponds.\*

**128.** The limits of a double integral are sometimes given in the form of a restriction upon the values which the independent variables  $x$  and  $y$  can simultaneously assume. Suppose, for example, that in the integration neither variable can become negative or exceed  $a$ , and that  $y$  cannot exceed  $x$ . If  $x$  and  $y$  are coordinates of a point in Fig. 21, the restriction is equivalent to saying that the point cannot be below the axis of  $x$ , to the right of the line  $x = a$  or above the line  $y = x$ . Therefore the area of integration is the triangle  $OAB$ . This may be expressed either by giving to  $y$  the limits zero and  $x$ , and then giving to  $x$  the limits zero and  $a$ ; or, reversing the order of integration, by giving to  $x$  the limits  $y$  and  $a$ , and then giving to  $y$  the limits zero and  $a$ . Thus the area of integration can be represented by either of the expressions

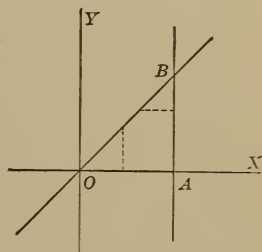


FIG. 21.

$$\int_0^a \int_0^x dy dx, \quad \text{or} \quad \int_0^a \int_y^a dx dy.$$

**129.** As an illustration, let us take the integral

$$U = \int_0^1 \int_0^x \sin^{-1} y \sqrt{1-x} \sqrt{x-y} dy dx.$$

The  $y$ -integration, which is here indicated as the first to be

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\* We have in the preceding articles supposed the first written integral sign to correspond to the last written differential when the pair of differentials is written last; but writers are not uniform in this respect.

performed, namely, the integration of  $\sqrt{(x-y)} \sin^{-1} y dy$  is impracticable. Noticing, however, that the  $x$ -integration in the given expression is of a known or integrable form, we change the order of integration, determining the new limits so as to represent the same area of integration. Since this area is that indicated in Fig. 21, when  $a = 1$ , we thus obtain

$$U = \int_0^1 \sin^{-1} y dy \int_y^1 \sqrt{(1-x)(x-y)} dx.$$

The value of the  $x$ -integral is  $\frac{1}{8}\pi(1-y)^2$ ; hence

$$U = \frac{\pi}{8} \int_0^1 \sin^{-1} y (1-y)^2 dy.$$

Finally, integrating by parts and then putting  $y = \sin \theta$ ,

$$U = \frac{\pi}{24} \int_0^1 \frac{(1-y)^3 dy}{\sqrt{(1-y^2)}} = \frac{\pi}{24} \int_0^{\frac{\pi}{2}} (1 - \sin \theta)^3 d\theta = \frac{\pi}{24} \left[ \frac{5\pi}{4} - \frac{11}{3} \right].$$

### *Triple Integrals.*

**130.** An expression of the general form

$$\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} \phi(x, y, z) dz dy dx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is called a *triple integral*. In the first integration the limits  $z_1$  and  $z_2$  are in general functions of  $x$  and  $y$ . In the next, the limits  $y_1$  and  $y_2$  are in general functions of  $x$ ; and in the final integration the limits are constants.

It is readily seen that the triple integral, like the double integral considered in Art. 122, is the limit (where  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  diminish without limit) of the result of a corresponding summation. Accordingly  $\phi(x, y, z) dz dy dx$  is called *the element* of the integral (1).

Consider now the triple integral with the same limits, but having the simpler element  $dz\,dy\,dx$ ; the  $z$ -integration can here be effected at once, and we have

$$\int_a^b \int_{x_1}^{y_2} \int_{z_1}^{z_2} dz\,dy\,dx = \int_a^b \int_{x_1}^{y_2} (z_2 - z_1)\,dy\,dx. \quad (2)$$

Now supposing  $x$ ,  $y$  and  $z$  to be rectangular coordinates in space, the last expression represents the difference between two solids defined as in Art. 126; that is, the triple integral in equation (2) represents *the volume included between the surfaces  $z = z_1$ ,  $z = z_2$ , and the cylindrical surface whose section in the plane of  $xy$  is the contour determined by the limits for  $y$  and  $x$ .*

**131.** When the volume is completely enclosed by a surface defined by a single equation or relation between  $x$ ,  $y$  and  $z$ , the case is analogous to that of the area discussed in Art. 124;  $z_1$  and  $z_2$  will be two values of  $z$  corresponding to the same values of  $x$  and  $y$ , and the limits for  $x$  and  $y$  must be determined by the area within which the value of  $z_2 - z_1$  is real. This area is the section with the plane of  $xy$  of a cylinder which circumscribes the given volume.

In this case it, is plain that we may perform the integrations in any order, provided we properly determine the limits. Considering the corresponding process of summation, we may be said in equation (2) to have summed those of the *ultimate elements* which have common values of  $x$  and  $y$  into the *linear element*  $(z_2 - z_1)\,dy\,dx$ . If the integration for  $y$  is effected next, we combine such of these last elements as have a common value of  $x$  into the *areal element*  $A_x\,dx$  which is often called a *lamina*. Thus each of the formulæ in Art. 118 is the result of performing two of the integrations in the general expression for a volume, namely,

$$V = \iiint dx\,dy\,dz.$$

In any case where the result of two integrations (that is, the area of a section) is known, we may of course take advantage of this and pass at once to the simple integral, as in Art. 118, where in finding the volume of the ellipsoid we regarded the expression for the area of an ellipse as known.

### Examples X.

1. Find the volume cut from a right circular cylinder whose radius is  $a$ , by a plane passing through the centre of the base and making the angle  $\alpha$  with the plane of the base.

$$\frac{2}{3}a^3 \tan \alpha.$$

2. Show that the integral of  $x \, dx \, dy$  over any area symmetrical to the axis of  $y$  vanishes. Interpret the result geometrically, and apply to find the integral of  $(c + mx + ny) \, dx \, dy$  over the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2. \quad \pi abc.$$

3. Show that the volume between the surface

$$z^n = a^2 x^2 + b^2 y^2$$

and any plane parallel to the plane of  $xy$  is equal to the circumscribing cylinder divided by  $n + 1$ .

4. Find the volume enclosed by the surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1. \quad \frac{8\pi abc}{5}.$$

5. A moving straight line, which is always perpendicular to a fixed straight line through which it passes, passes also through the circumference of a circle whose radius is  $a$ , in a plane parallel to the fixed straight line and at a distance  $b$  from it; find the volume enclosed by the surface generated and the circle.

$$\frac{\pi a^2 b}{2}.$$

6. A cylinder cuts the plane of  $xy$  in an ellipse whose semi-axes are  $a$  and  $b$ , and the plane of  $xz$  in an ellipse whose semi-axes are  $a$  and  $c$ , the elements of the cylinder being parallel to the plane of  $yz$ ; find the volume of the portion bounded by the semi-ellipses and the surface.

$$\frac{2}{3}abc.$$

7. Find the volume enclosed by the surface

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}$$

and the plane  $x = a$ .

$$\frac{abc}{2}.$$

8. Find the volume enclosed by the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Find  $A_z$  as in Art. 107.

$$\frac{4\pi a^3}{35}.$$

9. Find the volume between the coordinate planes and the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1.$$

$$\frac{abc}{90}.$$

10. Find the volume cut from the paraboloid of revolution

$$y^2 + z^2 = 4ax$$

by the right circular cylinder

$$x^2 + y^2 = 2ax,$$

whose axis intersects the axis of the paraboloid perpendicularly at the focus, and whose surface passes through the vertex.

$$2\pi a^3 + \frac{16a^3}{3}.$$

11. Find the volume cut from a sphere whose radius is
- $a$
- by a right circular cylinder whose radius is
- $b$
- , and whose axis passes through the centre of the sphere.

$$\frac{4\pi}{3} \left[ a^3 - (a^2 - b^2)^{\frac{3}{2}} \right].$$

12. Prove that the volume generated by the revolution about the axis of
- $y$
- of the area between the equilateral hyperbola

$$x^2 - y^2 = a^2$$

and the double ordinate  $2y$ , is equal to the sphere of radius  $y$ . [Ex. 11 shows that the circle has the same property.]



## XI.

*The Polar Element of Area.*

**132.** When polar coordinates are used, if concentric circles be drawn, corresponding to values of the radius vector  $r$  having the common difference  $\Delta r$ , and then straight lines through the pole corresponding to values of the angular coordinate  $\theta$  having the uniform difference  $\Delta\theta$ , any given portion of the plane may be divided, as in Fig. 22, into small areas. The value of any one of these is readily seen to be  $\Delta r \cdot r \Delta\theta$ , where  $r$  has a value midway between the greatest and least values of  $r$  in the area in question. It follows that the result

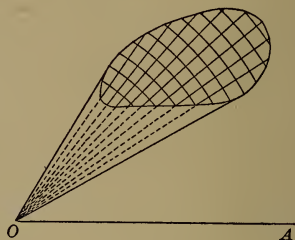


FIG. 22.

$$\Sigma \Sigma r \Delta r \Delta \theta$$

of a double summation of this element between proper limits will give an approximate expression for the given area. Hence the double integral

$$A = \iint r \, dr \, d\theta, \dots \dots \dots (I)$$

which (compare Art. 122) is the limit of this expression when  $\Delta r$  and  $\Delta\theta$  are both diminished without limit, is the exact expression for the given area.

**133.** The differential expression  $r \, dr \, d\theta$ , or *polar element of area*, is the product of the differentials  $dr$  and  $r \, d\theta$ , which correspond to the mutually rectangular dimensions  $\Delta r$  and  $r \Delta\theta$  of the element of summation. These factors are the differentials of the motions of the point  $(r, \theta)$ , respectively, when  $r$  alone



varies and when  $\theta$  alone varies. It is obvious that the element of area for any system of coordinates can in like manner be shown to be the product of the corresponding differential motions, provided only that these motions are *at right angles to one another*.\* The element is in such a case said to be *ultimately* a rectangle.

The formula used in section VIII is in fact the result of performing the integration for  $r$  in formula (1) above. Thus when the pole is outside of the given area, as in Fig. 22, we have

$$A = \iint_{r_1}^{r_2} r dr d\theta = \frac{1}{2} \int (r_2^2 - r_1^2) d\theta,$$

the formula of Art. 110. The limits for  $\theta$  are now the values which make  $r_2 = r_1$ ; that is, in the case of a continuous curve, the values for which the radius vector is tangent to the curve. So also when, as in the example of Art. 109, the pole is on the curve (so that  $r_1 = 0$  for all values of  $\theta$ ), the limits for  $\theta$  correspond to tangents to the curve. But, when the pole is within the curve, we assume the lower limit zero to avoid negative values of  $r$ , and then make  $\theta$  vary through a complete revolution, that is, from 0 to  $2\pi$ .

### *Cylindrical Coordinates.*

**134.** In determining the volume of a solid, it is sometimes convenient to use the polar coordinates  $r$  and  $\theta$  in place of  $x$  and  $y$ , while retaining the third coordinate  $z$  perpendicular to the plane of  $r\theta$ . The system  $r, \theta, z$  is sometimes called that of *cylindrical coordinates*, because the locus of  $r = \text{a constant}$

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\* In other words, whenever the loci of constant values of one coordinate cut orthogonally the like loci for the other coordinate; as in the case of the coordinates latitude and longitude on a spherical surface, or on a map made on any system in which the representatives of the parallels and meridians cut at right angles.

is the surface of a right circular cylinder. The element of volume in this system is obviously the product of  $dz$  and the polar element of area, that is,

$$r dr d\theta dz,$$

which has the ultimate form of a rectangular parallelepiped.

When this element is used in finding the volume of a solid bounded in part by a cylindrical surface of which the elements are parallel to the axis of  $z$ , like that represented in Fig. 20, Art. 126, the integration for  $z$  is naturally performed first, its limits being the values of  $z$  in terms of  $r$  and  $\theta$  given by the equations of the surfaces which cut the cylindrical surface.

**135.** For example, let us find the volume cut from a sphere of radius  $a$  by a right circular cylinder having a radius of the sphere for one of its diameters. Taking the centre of the sphere as origin, the diameter of the cylinder as initial line, and the axis of  $z$  parallel to the axis of the cylinder, we have, for the equation of the sphere, in these coordinates

$$z^2 + r^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, for that of the cylinder,

$$r = a \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The limits for  $z$  are now taken from equation (1), those of  $r$  are zero and that given by equation (2), and those of  $\theta$  are  $\pm \frac{1}{2}\pi$  which make  $r$  in equation (2) vanish. Hence we have for the required volume

$$\begin{aligned} V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a \cos \theta} r dr \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} dz \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a \cos \theta} (a^2 - r^2)^{\frac{1}{2}} r dr = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left[ -\frac{1}{3}(a^2 - r^2)^{\frac{3}{2}} \right]_0^{a \cos \theta}. \end{aligned}$$

From the symmetry of the solid it is apparent that we may take the limits for  $\theta$  to be 0 and  $\frac{1}{2}\pi$ , and double the result.\* Thus

$$V = \frac{4a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^3 \theta) d\theta = \frac{2\pi a^3}{3} - \frac{8a^3}{9}.$$

### *Solids of Revolution with Polar Coordinates.*

**136.** The volume of the solid produced by the revolution about the initial line of a curve given in polar coordinates may be expressed by a double integral of which the element is derived from the polar element of area. In this revolution, every point  $P$  of the polar element of summation,  $r \Delta r \Delta \theta$ , constructed in Fig. 23, describes a circle whose radius is  $PR = r \sin \theta$ , and moves always in a direction perpendicular to the plane of the element. Therefore the volume described by the element is equal to its area multiplied by  $2\pi r \sin \theta$ , where  $r$  and  $\theta$  belong to some point within the element. Since the distinction between all such points disappears at the limit, we have for the element of volume

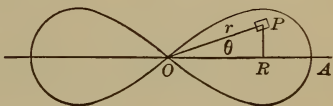


FIG. 23.

$$2\pi r^2 \sin \theta dr d\theta;$$

and the integral of this expression over the given area is the required volume.

\* In this example, the result, which would otherwise have been written in

the form  $V = \frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^3 \theta) d\theta$ , would not have been correct, because the

$(a^2 - r^2)^{\frac{3}{2}}$  which occurs in the integration stands for the *positive* value of the radical, whereas in the fourth quadrant  $a \sin \theta$ , which we have put for it, stands for the *negative* value.

137. For example, the curve in Fig. 23 being the lemniscata

$$r_1^2 = a^2 \cos 2\theta,$$

the volume generated by the right-hand loop, or rather by the half-loop in the first quadrant, is

$$V = 2\pi \int_0^{\frac{\pi}{4}} \int_0^{r_1} r^2 \sin \theta dr d\theta = \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \sin \theta d\theta.$$

Transforming, by putting  $x = \cos \theta$ , since

$$\cos 2\theta = 2 \cos^2 \theta - 1,$$

$$V = \frac{2\pi a^3}{3} \int_{\frac{1}{\sqrt{2}}}^1 (2x^2 - 1)^{\frac{3}{2}} dx;$$

and again, by putting  $x\sqrt{2} = \sec \phi$ ,

$$V = \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \tan^4 \phi \sec \phi d\phi = \frac{\pi a^3 \sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \frac{\sin^4 \phi}{\cos^5 \phi} d\phi.$$

Using now the formula of reduction (6), Art. 70, we find

$$\int \frac{\sin^4 \phi}{\cos^5 \phi} d\phi = \frac{\sin^3 \phi}{4 \cos^4 \phi} - \frac{3 \sin \phi}{8 \cos^2 \phi} + \frac{3}{8} \log \frac{1 + \sin \phi}{\cos \phi};$$

whence, applying the limits,

$$V = \frac{\pi a^3 \sqrt{2}}{3} \left[ \frac{3}{8} \log_e (\sqrt{2} + 1) - \frac{\sqrt{2}}{8} \right] = 0.228 \times a^3.$$

### *Polar Coordinates in Space.*

138. The double integral employed in the preceding article is in fact an application of *the polar system of coordinates in space*. In this system, a point is determined by polar coordinates in a plane which passes through a fixed axis and makes the diedral angle  $\phi$  with a fixed plane of reference passing through the axis. In Art. 136, the fixed axis is the initial line

$OA$ , and taking the plane of the paper as the plane of reference, the third coordinate  $\phi$  is the angle described by the upper half-plane in the revolution of the figure. The differential of the motion of  $P$  when  $\phi$  varies is  $r \sin \theta d\phi$ , and this motion is at right angles to the plane of the differentials  $dr$  and  $r d\theta$ ; hence the ultimate element of volume is  $r^2 \sin \theta dr d\theta d\phi$ . But in the case of the solid of revolution the limits of the  $\phi$ -integration are 0 and  $2\pi$ , so that the result of this integration is  $2\pi r^2 \sin \theta dr d\theta$ , the element of the double integral derived in Art. 136.

**139.** In the equations of transformation connecting these coordinates with the rectangular system, it is usual to take the axis of  $z$  as that about which the angle  $\phi$  is described, and the plane  $xz$  as that for which  $\phi = 0$ . Then in the plane  $ZOR$ , Fig. 24, the line  $OR$  is usually taken as the initial line for the polar coordinates  $\rho$  and  $\theta$ , so that the three coordinates of  $P$  are

$$\rho = OP, \quad \theta = POR, \quad \phi = AOR.$$

Thus  $\rho$  in this system is the distance of the point  $P$  from a fixed pole, and  $\theta$  and  $\phi$  correspond to the spherical coordinates latitude and longitude on a sphere whose radius is  $\rho$ . Denote the radius of the circle of latitude  $BP$  by  $r$ ; then

$$r = \rho \cos \theta,$$

and  $r$  and  $\phi$  are the polar coordinates of the projection of  $P$  in the plane of  $xy$ . Hence the formulæ connecting the rectangular coordinates  $x, y, z$  with  $\rho, \theta$  and  $\phi$  are

$$x = r \cos \phi = \rho \cos \theta \cos \phi,$$

$$y = r \sin \phi = \rho \cos \theta \sin \phi,$$

$$z = \rho \sin \theta.$$

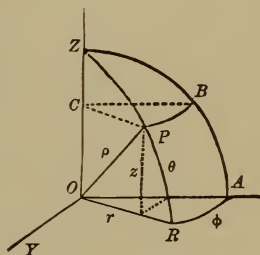


FIG. 24.

The differentials of the motions of  $P$  when one of the coordinates  $\rho$ ,  $\theta$ , and  $\phi$  varies, the other two remaining constant, are, respectively,

$$d\rho, \quad \rho d\theta, \quad \text{and} \quad \rho \cos \theta d\phi;$$

hence the element of volume is their product,

$$\rho^2 \cos \theta d\rho d\theta d\phi,$$

in which the factor  $\cos \theta$  occurs, in place of the factor  $\sin \theta$  which appears in the element derived in the preceding article, because the axis of rotation is now perpendicular to the initial line.

In the case of a sphere whose centre is at the pole, all the limits are constant, and we have

$$V = \int_0^a \rho^2 d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^3}{3}.$$

### *Volumes in general.*

**140.** We have seen in Art. 123 that the boundary of the area expressed by a double integral may consist in part of lines whose equations contain only one of the variables, namely, that for which the final integration takes place. But, as explained in Art. 125, in the general case, it is necessary to regard the given area as made up of positive or negative parts of the kind just mentioned. This is done by drawing the loci of constant values of the final variable through the intersections of the lines forming the boundary, or else tangent to them as in Fig. 19. These parts are then expressed by separate integrals.

So also we have seen in Art. 126 that the volume expressed by a triple integral may be bounded in part by a surface whose equation contains only two of the variables, namely, those for which the last two integrations take place. But, in the general case, it is necessary to separate the volume into parts, by means

of such surfaces passed through the lines of intersections of the bounding surfaces, or edges of the given solid.

141. Figs. 19 and 20 illustrate this for rectangular coordinates, but similar considerations apply to any other system, and enable us to decide whether it is possible to express a given volume by a single integral. For example, let it be required to find the volume common to the solid of revolution produced by the half-cardioid  $OAB$ , Fig. 25, revolving about its axis  $OB$ , and the sphere whose centre is at the pole and whose radius is  $OC = c$ . The volume is the result of integrating the element

$$2\pi r^2 \sin \theta dr d\theta$$

(found as in Art. 136) over the area  $OAC$ . For this area of integration,  $\theta = \pi$  and  $r = 0$  give one constant limit for each variable, and the others are determined by the equation of the arc  $OA$  of the cardioid

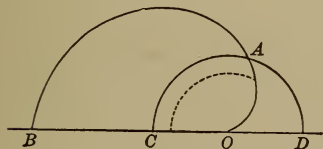


FIG. 25.

$$r = a(1 - \cos \theta), \quad (1)$$

and that of the circular arc  $AC$ ,

$$r = c. \quad (2)$$

Since equation (1) contains both variables, while equation (2) contains  $r$  only, we can, by integrating first for  $\theta$  (and using equation (1) for one of its limits), express the required magnitude by a single integral; thus,

$$V = 2\pi \int_0^c \int_0^\pi r^2 \sin \theta d\theta dr = 2\pi \int_0^c r^2 (1 + \cos \theta) dr.$$

Substituting the value of  $\cos \theta$  from equation (1),

$$V = 2\pi \int_0^c r^2 \left( 2 - \frac{r}{a} \right) dr = \frac{\pi c^3}{6a} (8a - 3c).$$



If in this example we integrate first for  $r$ , it becomes necessary to find  $\theta_1$ , the value of  $\theta$  for the point of intersection  $A$ , and, regarding the area of integration as separated into two parts by the radius vector  $OA$ , to form two integrals, in one of which the upper limit for  $r$  is taken from equation (1), and the limits for  $\theta$  are  $0^*$  and  $\theta_1$ ; while in the other  $r$  is taken from equation (2), and the limits for  $\theta$  are  $\theta_1$  and  $\pi$ .

### Examples XI.

1. The paraboloid of revolution

$$x^2 + y^2 = cz$$

is pierced by the right circular cylinder

$$x^2 + y^2 = ax,$$

whose diameter is  $a$ , and whose surface contains the axis of the paraboloid; find the volume between the plane of  $xy$  and the surfaces of the paraboloid and of the cylinder.

$$\frac{3\pi a^4}{32c}.$$

2. Find the volume cut from a sphere whose radius is  $a$  by the cylinder whose base is the curve

$$r = a \cos 3\theta. \qquad \frac{2a^3\pi}{3} - \frac{8a^3}{9}.$$

3. Find the volume cut from a sphere whose radius is  $a$  by the cylinder whose base is the curve

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta,$$

supposing  $b < a$ .

$$\frac{4\pi a^3}{3} - \frac{16}{9}(a^2 - b^2)^{\frac{3}{2}}.$$

---

\* This limit also is in fact determined by the intersection of two sides of the area of integration, namely, that of the curve (1) and  $r = 0$  the vanishing inner edge of the area.



4. A right cone, the radius of whose base is  $a$  and whose altitude is  $b$ , is pierced by a cylinder whose base is a circle having for diameter a radius of the base of the cone; find the volume common to the cone and the cylinder.

$$\frac{ba^2}{36}(9\pi - 16).$$

5. The axis of a right cone whose semi-vertical angle is  $\alpha$  coincides with a diameter of the sphere whose radius is  $a$ , the vertex being on the surface of the sphere; find the volume of the portion of the sphere which is outside of the cone.

$$\frac{4\pi a^3 \cos^4 \alpha}{3}.$$

6. Find the volume produced by the revolution of the lemniscata

$$r^2 = a^2 \cos 2\theta$$

about a perpendicular to the initial line.

$$\frac{\pi^2 a^3 \sqrt{2}}{8}.$$

7. Find the volumes generated by the revolution of the large loop and by one of the small loops of the curve

$$r = a \cos \theta \cos 2\theta$$

about a perpendicular to the initial line.

$$\frac{\pi^2 a^3}{16} + \frac{\pi a^3}{5}; \quad \frac{\pi^2 a^3}{32} - \frac{\pi a^3}{10}.$$

8. What is the volume of the cavity which will just permit the cardioid

$$r = a(1 - \cos \theta)$$

to rotate about a line on its plane perpendicular to the initial line?

$$\frac{16 + 5\pi}{4} \pi a^3.$$

9. Find the whole volume enclosed by the surface

$$(x^2 + y^2 + z^2)^3 = a^3 xyz.$$

*Transform to the coordinates  $\rho$ ,  $\phi$ ,  $\theta$ , and show that the solid consists of four equal detached parts.*

$$\frac{a^3}{6}.$$

10. Find the volume of that part of the sphere

$$x^2 + y^2 + z^2 = a(x + y + z)$$

for which all three coordinates are positive.

$$\frac{2\pi + 1}{4}a^3.$$

11. Find the volume between the surface generated by the revolution of the cardioid

$$r = a(1 - \cos \theta)$$

about the initial line and the plane which touches this surface in a circle.

$$\frac{\pi a^3}{192}.$$

## XII.

### *Rectification of Plane Curves.*

142. A curve is said to be *rectified* when its length is determined, the unit of measure to which it is referred being a right line.

It is shown in Diff. Calc., Art. 314 [Abridged Ed., Art. 164], that, if  $s$  denotes the length of the arc of a curve given in rectangular coordinates, we shall have

$$ds = \sqrt{dx^2 + dy^2}.$$

If the abscissas of the extremities of the arc are known,  $s$  is found by substituting for  $dy$  in this expression its value in terms of  $x$  and  $dx$ , and integrating the result between the given values of  $x$  as limits. Thus, to express the arc measured from the vertex of the semi-cubical parabola

$$ay^2 = x^3$$

in terms of the abscissa of its other extremity, we derive, from the equation of the curve,

$$dy = \frac{3\sqrt{x} dx}{2\sqrt{a}},$$

whence

$$ds = \frac{\sqrt{9x + 4a}}{2\sqrt{a}} dx.$$

Integrating,

$$\begin{aligned} s &= \frac{1}{2\sqrt{a}} \int_0 \sqrt{9x + 4a} dx \\ &= \frac{1}{27\sqrt{a}} (9x + 4a)^{\frac{3}{2}} - \frac{8a}{27}. \end{aligned}$$

**143.** When  $x$  and  $y$  are most conveniently expressed as functions of a third variable, the expression for  $ds$  in terms of this variable may be used. For example, in the case of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

we put, as in Art. 107,

$$x = a \sin^3 \psi,$$

$$y = b \cos^3 \psi;$$

whence

$$dx = 3a \sin^2 \psi \cos \psi d\psi,$$

$$dy = -3b \cos^2 \psi \sin \psi d\psi;$$

and therefore

$$\begin{aligned} ds &= 3 \sin \psi \cos \psi d\psi \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi} \\ &= \frac{3}{2} \sqrt{(a^2 - b^2) \sin^2 \psi + b^2} d(\sin^2 \psi). \end{aligned}$$

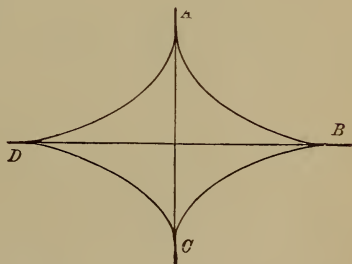


FIG. 26.

The points  $A$  and  $B$ , Fig. 26, correspond to  $\psi = 0$  and  $\psi = \frac{1}{2}\pi$  respectively; hence, integrating, we have

$$\text{arc } AB = \frac{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}}}{a^2 - b^2} \Bigg|_0^{\frac{\pi}{2}} = \frac{a^3 - b^3}{a^2 - b^2} = \frac{a^2 + ab + b^2}{a + b}.$$

### *Change of the Sign of $ds$ .*

**144.** In the example above,  $x$  and  $y$  being one-valued functions of  $\psi$ , every value of  $\psi$  corresponds to a definite point of the curve; and, as  $\psi$  varies from 0 to  $2\pi$ , the point  $(x, y)$  describes the whole curve in the direction  $ABCD$ . But it will be noticed that,  $d\psi$  remaining positive, the value of  $ds$  becomes zero and changes sign when  $\psi$  passes through either of the values 0,  $\frac{1}{2}\pi$ ,  $\pi$ , or  $\frac{3}{2}\pi$ . This corresponds geometrically to the fact that the point  $(x, y)$  stops and reverses the direction of its motion, forming a *stationary point* or *cusp* at each of the points  $A$ ,  $B$ ,  $C$ , and  $D$ , as shown in Fig. 26. Such points are thus indicated by a change of sign in  $ds$ , and the arcs between them must be considered separately, because the corresponding definite integrals have different signs.

### *Polar Coordinates.*

**145.** It is proved in Diff. Calc., Art. 317 [Abridged Ed., Art. 167], that when the curve is given in polar coordinates

$$ds = \sqrt{(dr^2 + r^2 d\theta^2)}.$$

This is usually expressed in terms of  $\theta$ . For example, the equation of the cardioid is

$$r = a(1 - \cos \theta) = 2a \sin^2 \frac{1}{2}\theta;$$

whence  $dr = 2a \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta d\theta$ ,

and by substitution

$$ds = 2a \sin \frac{1}{2}\theta d\theta.$$

The limits for the whole perimeter of the curve are 0 and  $2\pi$ , and  $ds$  remains positive for the whole interval. Therefore

$$s = 2a \int_0^{2\pi} \sin \frac{\theta}{2} d\theta = -4a \cos \frac{\theta}{2} \Big|_0^{2\pi} = 8a.$$

### *Rectification of Curves of Double Curvature.*

**146.** Let  $\sigma$  denote the length of the arc of a *curve of double curvature*; that is, one which does not lie in a plane, and suppose the curve to be referred to rectangular coordinates  $x$ ,  $y$  and  $z$ . If at any point of the curve the differentials of the coordinates be drawn in the directions of their respective axes, a rectangular parallelopiped will be formed, whose sides are  $dx$ ,  $dy$  and  $dz$ , and whose diagonal is  $d\sigma$ . Hence

$$d\sigma = \sqrt{(dx^2 + dy^2 + dz^2)}.$$

The curve is determined by means of two equations connecting  $x$ ,  $y$  and  $z$ , one of which usually expresses the value of  $y$  in terms of  $x$ , and the other that of  $z$  in terms of  $x$ . We can then express  $d\sigma$  in terms of  $x$  and  $dx$ .

If the given equations contain all the variables, equations of the required form may be obtained by elimination.

**147.** An equation containing the two variables  $x$  and  $y$  only is evidently the equation of the *projection upon the plane of  $xy$*  of a curve traced upon the surface determined by the other equation. Let  $s$  denote the length of this projection; then, since  $ds^2 = dx^2 + dy^2$ ,

$$d\sigma = \sqrt{(ds^2 + dz^2)},$$

in which  $ds$  may, if convenient, be expressed in polar coordinates; thus,

$$d\sigma = \sqrt{(dr^2 + r^2 d\theta^2 + dz^2)}.$$

148. As an illustration, let us use this formula to determine the length of the loxodromic curve from the equation of the sphere,

$$x^2 + y^2 + z^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

upon which it is traced, and its projection upon the plane of the equator, of which the equation is

$$2a = \sqrt{(x^2 + y^2)} \left( \varepsilon^{n \tan^{-1} \frac{y}{x}} + \varepsilon^{-n \tan^{-1} \frac{y}{x}} \right),$$

or in polar coordinates

$$2a = r (\varepsilon^{n\theta} + \varepsilon^{-n\theta}). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) is equivalent to

$$r^2 + z^2 = a^2;$$

and, denoting the latitude of the projected point by  $\phi$ , this gives

$$z = a \sin \phi, \quad r = a \cos \phi. \quad . \quad . \quad . \quad (3)$$

In order to express  $d\theta$  in terms of  $\phi$ , we substitute the value of  $r$  in (2); whence

$$\varepsilon^{n\theta} + \varepsilon^{-n\theta} = 2 \sec \phi, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and by differentiation

$$\varepsilon^{n\theta} - \varepsilon^{-n\theta} = \frac{2}{n} \sec \phi \tan \phi \frac{d\phi}{d\theta}. \quad . \quad . \quad . \quad . \quad (5)$$

Squaring and subtracting equation (5) from equation (4),

$$4 = \frac{4 \sec^2 \phi}{n^2} \left[ n^2 - \tan^2 \phi \frac{d\phi^2}{d\theta^2} \right],$$

which reduces to

$$d\theta^2 = \frac{\sec^2 \phi d\phi^2}{n^2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From equations (3) and (6)

$$r^2 d\theta^2 = \frac{a^2}{n^2} d\phi^2,$$

$$dr^2 = a^2 \sin^2 \phi d\phi^2,$$

$$dz^2 = a^2 \cos^2 \phi d\phi^2;$$

whence substituting in the value of  $d\sigma$  (p. 177)

$$d\sigma = a \sqrt{1 + \frac{1}{n^2}} d\phi.$$

Integrating,

$$\sigma = a \frac{\sqrt{(n^2 + 1)}}{n} \int_{\alpha}^{\beta} d\phi = a \frac{\sqrt{(n^2 + 1)}}{n} (\beta - \alpha),$$

where  $\alpha$  and  $\beta$  denote the latitudes of the extremities of the arc.

### Examples XII.

1. Find the length of an arc measured from the vertex of the catenary

$$y = \frac{c}{2} \left( \varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}} \right),$$

and show that the area between the coordinate axes and any arc is proportional to the arc.

$$s = \frac{c}{2} \left( \varepsilon^{\frac{x}{c}} - \varepsilon^{-\frac{x}{c}} \right).$$

$$A = cs.$$

2. Find the length of an arc measured from the vertex of the parabola

$$y^2 = 4ax.$$

$$\sqrt{ax + x^2} + a \log \frac{\sqrt{x} + \sqrt{x+a}}{\sqrt{a}}.$$

3. Find the length of the curve

$$y = \frac{e^x + 1}{e^x - 1},$$

between the points whose abscissas are  $a$  and  $b$ .

$$\log \frac{e^{2b} - 1}{e^{2a} - 1} + a - b.$$

4. Find the length, measured from the origin, of the curve

$$y = a \log \frac{a^2 - x^2}{a^2}.$$

$$a \log \frac{a + x}{a - x} - x.$$

5. Given the differential equation of the tractrix,

$$\frac{dy}{dx} = - \frac{y}{\sqrt{(a^2 - y^2)}},$$

and, assuming  $(0, a)$  to be a point of the curve, find the value of  $s$  as measured from this point, and also the value of  $x$  in terms of  $y$ ; that is, find the rectangular equation of the curve.

$$s = a \log \frac{y}{a}.$$

$$x = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

6. Find the length of one branch of the cycloid

$$x = a(\psi - \sin \psi),$$

$$y = a(1 - \cos \psi).$$

$8a.$

7. When the cycloid is referred to its vertex, the equations being

$$x = a(1 - \cos \psi),$$

$$y = a(\psi + \sin \psi),$$

prove that

$$s = \sqrt{(8ax)}.$$



8. Find the length from the point  $(a, 0)$  of the curve

$$x = 2a \cos \psi - a \cos 2\psi,$$

$$y = 2a \sin \psi - a \sin 2\psi.$$

$$8a (1 - \cos \frac{1}{2} \psi).$$

9. Show that the curve,

$$x = 3a \cos \psi - 2a \cos^3 \psi,$$

$$y = 2a \sin^3 \psi,$$

has cusps at the points given by  $\psi = 0$  and  $\psi = \pi$ ; and find the whole length of the curve.

12a.

10. Find the length of the arc of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

between the points where it touches the axes.

$$\frac{a^3 + b^3}{a^2 + b^2} + \frac{a^2 b^2}{(a^2 + b^2)^{\frac{3}{2}}} \log \frac{[\sqrt{(a^2 + b^2)} + a][\sqrt{(a^2 + b^2)} + b]}{ab}.$$

11. Show that the curve

$$x = 2a \cos^2 \theta (3 - 2 \cos^2 \theta),$$

$$y = 4a \sin \theta \cos^3 \theta$$

has three cusps, and that the length of each branch is  $\frac{8a}{3}$ .

12. Find the length of the arc between the points at which the curve

$$x = a \cos^2 \theta \cos 2\theta,$$

$$y = a \sin^2 \theta \sin 2\theta$$

cuts the axes.

$$\frac{2 + \sqrt{2}}{3} a.$$

13. Show that the curve

$$\begin{aligned}x &= a \cos \psi (1 + \sin^2 \psi), \\y &= a \sin \psi \cos^2 \psi\end{aligned}$$

is symmetrical to the axes, and find the length of the arcs between the cusps.

$$a \left( \sqrt{2} - \sin^{-1} \frac{1}{\sqrt{3}} \right);$$

$$a \left( \sqrt{2} + \cos^{-1} \frac{1}{\sqrt{3}} \right).$$

14. Find the length of one branch of the epicycloid

$$x = (a + b) \cos \psi - b \cos \frac{a+b}{b} \psi,$$

$$y = (a + b) \sin \psi - b \sin \frac{a+b}{b} \psi.$$

$$\frac{8b(a+b)}{a}.$$

15. Show that the curve

$$\begin{aligned}x &= 9a \sin \psi - 4a \sin^3 \psi, \\y &= -3a \cos \psi + 4a \cos^3 \psi\end{aligned}$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs,  $(\alpha)$  between the double points,  $(\beta)$  between a double point and a cusp, and  $(\gamma)$  the arc connecting two cusps, and not passing through the double points.

$$(\alpha), a(\pi + 3\sqrt{3});$$

$$(\beta), \frac{\pi a}{2};$$

$$(\gamma), a(3\sqrt{3} - \pi).$$

16. Find the whole length of the curve

$$\begin{aligned}x &= 3a \sin \psi - a \sin^3 \psi, \\y &= a \cos^3 \psi.\end{aligned}$$

$$3\pi a.$$

17. Find the length, measured from the pole, of any arc of the equiangular spiral

$$r = a\varepsilon^{n\theta},$$

in which  $n = \cot \alpha$ .

$$r \sec \alpha.$$

18. Prove by integration that the arc subtending the angle  $\theta$  at the circumference in a circle whose radius is  $a$ , is  $2a\theta$ .

19. Find the length, measured from the origin, of the curve defined by the equations

$$y = \frac{x^2}{2a},$$

$$z = \frac{x^3}{6a^2}.$$

$$x + \frac{x^3}{6a^2}.$$

20. Find the length, measured from the origin, of the intersection of the surfaces

$$y = 4n \sin x,$$

$$z = 2n^2 (2x + \sin 2x).$$

$$(4n^2 + 1)x + 2n^2 \sin 2x.$$

21. Find the length, measured from the origin, of the intersection of the cylindrical surfaces

$$(y - x)^2 = 4ax,$$

$$9a(z - x)^2 = 4x^3.$$

$$\frac{2x^{\frac{3}{2}}}{3\sqrt{a}} + 2\sqrt{ax} + x.$$

22. If upon the hyperbolic cylinder

$$\frac{y^2}{c^2} - \frac{z^2}{b^2} = 1,$$

a curve whose projection upon the plane of  $xy$  is the catenary

$$y = \frac{c}{2} \left( \varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}} \right)$$

be traced, prove that any arc of the curve bears to the corresponding arc of its projection the constant ratio  $\sqrt{b^2 + c^2} : c$ .

## XIII.

*Surfaces of Solids of Revolution.*

**149.** The surface of a solid of revolution may be generated by the circumference of the circular section made by a plane perpendicular to the axis of revolution. Thus in Fig. 27, the surface produced by the revolution of the curve  $AB$  about the axis of  $x$  is regarded as generated by the circumference  $PQ$ . The radius of this circumference is  $y$ , and its plane has a motion whose differential is  $dx$ , but every point in the circumference itself has a motion whose differential is  $ds$ ,  $s$  denoting an arc of the curve  $AB$ .

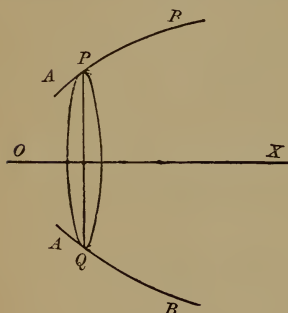


FIG. 27.

Hence, denoting the required surface by  $S$ , we have

$$dS = 2\pi y ds = 2\pi y \sqrt{(dx^2 + dy^2)}.$$

The value of  $dS$  must of course be expressed in terms of a single variable before integration.

**150.** For example, let us determine the area of the zone of spherical surface included between any two parallel planes. The radius of the sphere being  $a$ , the equation of the revolving curve is

$$x^2 + y^2 = a^2;$$

whence

$$y = \sqrt{(a^2 - x^2)},$$

$$dy = -\frac{x dx}{\sqrt{(a^2 - x^2)}},$$

$$ds = \frac{a dx}{\sqrt{(a^2 - x^2)}},$$

and

$$dS = 2\pi a dx;$$

therefore

$$S = 2\pi a \int dx = 2\pi a (x_2 - x_1).$$

Since  $x_2 - x_1$  is the distance between the parallel planes, the area of a zone is the product of its altitude by  $2\pi a$ , the circumference of a great circle, and the area of the whole surface of the sphere is  $4\pi a^2$ .

**151.** When the curve is given in polar coordinates, it is convenient to transform the expression for  $S$  to polar coordinates. Thus, if the curve revolves about the initial line,

$$S = 2\pi \int y \, ds = 2\pi \int r \sin \theta \, \sqrt{(dr^2 + r^2 d\theta^2)}.$$

For example, if the curve is the cardioid

$$r = 2a \sin^2 \frac{1}{2} \theta,$$

we find, as in Art. 145,

$$ds = 2a \sin \frac{1}{2} \theta \, d\theta.$$

Hence

$$\begin{aligned} S &= 16\pi a^2 \int_0^\pi \sin^4 \frac{1}{2} \theta \cos \frac{1}{2} \theta \, d\theta \\ &= \frac{32\pi a^2}{5} \sin^5 \frac{1}{2} \theta \Big|_0^\pi = \frac{32\pi a^2}{5}. \end{aligned}$$

### *Areas of Surfaces in General.*

**152.** Let a surface be referred to rectangular coordinates  $x$ ,  $y$  and  $z$ ; the projection of a given portion of the surface upon the plane of  $xy$  is a plane area determined by a given relation between  $x$  and  $y$ . We may take as the elements of the surface the portions which are projected upon the corresponding

elements of area in the plane of  $xy$ . If at a point within the element of surface, which is projected upon a given element  $\Delta x \Delta y$ , a tangent plane be passed, and if  $\gamma$  denote the inclination of this plane to the plane of  $xy$ , the area of the corresponding element *in the tangent plane* is

$$\sec \gamma \Delta x \Delta y.$$

The surface is evidently the limit of the sum of the elements in the tangent planes when  $\Delta x$  and  $\Delta y$  are indefinitely diminished. Now  $\sec \gamma$  is a function of the coordinates of the point of contact of the tangent plane; and since these coordinates are values of  $x$  and  $y$  which lie respectively between  $x$  and  $x + \Delta x$  and between  $y$  and  $y + \Delta y$ , it follows, as in Art. 122, that this limit is

$$S = \iint \sec \gamma \, dx \, dy.$$

**153.** The value of  $\sec \gamma$  may be derived by the following

method. Through the point  $P$  of the surface let planes be passed parallel to the coordinate planes, and let  $PD$ , and  $PE$ , Fig. 28, be the intersections of the tangent plane with the planes parallel to the planes of  $xz$  and  $yz$ . Then  $PD$  and  $PE$  are tangents at  $P$  to the sections of the surface made by these planes. The equations of these

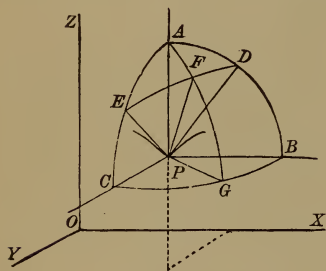


FIG. 28.

and  $x$  in turn as constants in the equation of the surface; therefore denoting the inclinations of these tangent lines to the plane of  $xy$  by  $\phi$  and  $\psi$ , we have

$$\tan \phi = \frac{dz}{dx}, \quad \text{and} \quad \tan \psi = \frac{dz}{dy},$$

in which  $\frac{dz}{dx}$  and  $\frac{dz}{dy}$  are partial derivatives derived from the equation of the surface.

If the planes be intersected by a spherical surface whose centre is  $P$ ,  $ADE$  is a spherical triangle right angled at  $A$ , whose sides are the complements of  $\phi$  and  $\psi$ . Moreover, if a plane perpendicular to the tangent plane  $PED$  be passed through  $AP$ , the angle  $FPG$  will be  $\gamma$ , and the perpendicular from the right angle to the base of the triangle the complement of  $\gamma$ .

Denoting the angle  $EAF$  by  $\theta$ , the formulæ for solving spherical right triangles give

$$\cos \theta = \frac{\tan \psi}{\tan \gamma}, \quad \text{and} \quad \sin \theta = \frac{\tan \phi}{\tan \gamma}.$$

Squaring and adding,

$$1 = \frac{\tan^2 \psi + \tan^2 \phi}{\tan^2 \gamma},$$

$$\text{or } \tan^2 \gamma = \tan^2 \psi + \tan^2 \phi;$$

$$\text{whence} \quad \sec^2 \gamma = 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2.$$

Substituting in the formula derived in Art. (152), we have

$$S = \iint \sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]} dx dy.$$

**154.** It is sometimes more convenient to employ the polar

element of the projected area. Thus the formula becomes

$$S = \iint \sec \gamma r dr d\theta,$$

where  $\sec \gamma$  has the same meaning as before.

For example, let it be required to find the area of the surface of a hemisphere intercepted by a right cylinder having a radius of the hemisphere for one of its diameters. From the equation of the sphere,

$$x^2 + y^2 + z^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we derive

$$\frac{dz}{dx} = -\frac{x}{z}, \quad \frac{dz}{dy} = -\frac{y}{z};$$

whence

$$\sec \gamma = \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} = \frac{a}{z};$$

therefore

$$S = a \iint \frac{r dr d\theta}{z},$$

the integration extending over the area of the circle

$$r = a \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since equation (1) is equivalent to

$$z^2 + r^2 = a^2,$$

$$S = a \int \int_r^{r_2} \frac{r dr}{\sqrt{a^2 - r^2}} d\theta = a \int [\sqrt{a^2 - r_1^2} - \sqrt{a^2 - r_2^2}] d\theta.$$



From (2) the limits for  $r$  are  $r_1 = 0$ , and  $r_2 = a \cos \theta$ , hence

$$S = a^2 \int (1 - \sin \theta) d\theta,$$

in which  $a \sin \theta$  is put for the *positive* quantity  $\sqrt{a^2 - r_2^2}$ . The limits for  $\theta$  are  $-\frac{1}{2}\pi$  and  $\frac{1}{2}\pi$ , but since  $\sin \theta$  is in this case to be regarded as invariable in sign, we must write

$$S = 2a^2 \int_0^{\frac{\pi}{2}} (1 - \sin \theta) d\theta = \pi a^2 - 2a^2.$$

If another cylinder be constructed, having the opposite radius of the hemisphere for diameter, the surface removed is  $2\pi a^2 - 4a^2$ , and the surface which remains is  $4a^2$ , a quantity commensurable with the square of the radius. This problem was proposed in 1692, in the form of an enigma, by Viviani, a Florentine mathematician.

### Examples XIII.

1. Find the surface of the paraboloid whose altitude is  $a$ , and the radius of whose base is  $b$ .

$$\frac{\pi b}{6a^2} [ (4a^2 + b^2)^{\frac{3}{2}} - b^3 ]$$

2. Prove that the surface generated by the arc of the catenary given in Ex. XII., 1, revolving about the axis of  $x$ , is equal to

$$\pi(cx + sy).$$

3. Find the whole surface of the oblate spheroid produced by the

revolution of an ellipse about its minor axis,  $a$  denoting the major,  $b$  the minor semi-axis, and  $e$  the excentricity,  $\frac{\sqrt{a^2 - b^2}}{a}$ .

$$2\pi a^2 + \pi \frac{b^2}{e} \log \frac{1+e}{1-e}.$$

4. Find the whole surface of the prolate spheroid produced by the revolution of the ellipse about its major axis, using the same notation as in Ex. 3.

$$2\pi b^2 + 2\pi ab \frac{\sin^{-1} e}{e}.$$

5. Find the surface generated by the cycloid

$$x = a(\psi - \sin \psi), \quad y = a(1 - \cos \psi)$$

revolving about its base.

$$\frac{64}{3} \pi a^2.$$

6. Find the surface generated when the cycloid revolves about the tangent at its vertex.

$$\frac{32}{3} \pi a^2.$$

7. Find the surface generated when the cycloid revolves about its axis.

$$8\pi a^2 \left( \pi - \frac{4}{3} \right).$$

8. Find the surface generated by the revolution of one branch of the tractrix (see Ex. XII, 5) about its asymptote.

$$2\pi a^2.$$

9. Find the surface generated by the revolution about the axis of  $x$  of the portion of the curve

$$y = \varepsilon^x,$$

which is on the left of the axis of  $y$ .

$$\pi[\sqrt{2} + \log(1 + \sqrt{2})].$$

10. Find the surface generated by the revolution about the axis of  $x$  of the arc between the points for which  $x = a$  and  $x = b$  in the hyperbola

$$xy = k^2.$$

$$\pi k^2 \left[ \log \frac{b^2 + \sqrt{k^4 + b^4}}{a^2 + \sqrt{k^4 + a^4}} + \frac{\sqrt{k^4 + a^4}}{a^2} - \frac{\sqrt{k^4 + b^4}}{b^2} \right].$$

11. Show that the surface of a cylinder whose generating lines are parallel to the axis of  $z$  is represented by the integral

$$S = \int z \, ds,$$

where  $s$  denotes the arc of the base in the plane of  $xy$ . Hence, deduce the surface cut from a right circular cylinder whose radius is  $a$ , by a plane passing through the centre and making the angle  $\alpha$  with the plane of the base.

$$2a^2 \tan \alpha.$$

12. Find the surface of that portion of the cylinder in the problem solved in Art. 154, which is within the hemisphere.

$$2a^2.$$

13. Find the surface of a circular spindle,  $a$  being the radius and  $2c$  the chord.

$$4\pi a \left[ c - \sqrt{a^2 - c^2} \sin^{-1} \frac{c}{a} \right].$$

## XIV.

*The Area generated by a Straight Line moving in any Manner in a Plane.*

155. If a straight line of indefinite length moves in any manner whatever in a plane, there is at each instant a point of the line about which it may be regarded as rotating. This point we shall call *the centre of rotation* for the instant. The rate of motion of every point of the line in a direction perpendicular to the line itself is at the instant the same as it would be if the line were rotating at the same angular rate about this point as a fixed centre.\* Hence it follows that the area generated by a definite portion of the line has at the instant the same rate as if the line were rotating about a fixed instead of a variable centre.

156. Suppose at first that the centre of rotation is on the generating line produced,  $\rho_1$  and  $\rho_2$  denoting the distances from the centre of the extremities of the generating line, and let  $\phi$  denote its inclination to a fixed line. By substitution in the general formula derived in Art. 110, we have

$$dA = \frac{1}{2}(\rho_2^2 - \rho_1^2) d\phi.$$

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\* Compare Diff. Calc., Art. 332 [Abridged Ed., Art. 176], where the moving line is the normal to a given curve, and the centre of rotation is the centre of curvature of the given curve. If the line is moving without change of direction, the centre is of course at an infinite distance.

When the line is regarded as forming a part of a rigidly connected system in motion, its centre of rotation is the foot of a perpendicular dropped upon it from the *instantaneous centre* of the motion of the system. Thus, if the tangent and normal in the illustration cited are rigidly connected, the centre of curvature,  $C$ , is the *instantaneous centre* of the motion of the system, and the point of contact,  $P$ , is the centre of rotation for the tangent line.

*Applications.*

**157.** The area between a curve and its evolute may be generated by the radius of curvature  $\rho$ , whose inclination to the axis of  $x$  is  $\phi + \frac{1}{2}\pi$ , in which  $\phi$  denotes the inclination of the tangent line. Since the centre of rotation is one extremity of the generating line  $\rho$ , the differential of this area is found by substituting in the general expression  $\rho_1 = 0$  and  $\rho_2 = \rho$ . Hence when  $\rho$  is expressed in terms of  $\phi$ ,

$$A = \frac{1}{2} \int \rho^2 d\phi$$

expresses the area between an arc of a given curve, its evolute, and the radii of curvature of its extremities, the limits being the values of  $\phi$  at the ends of the given arc.

**158.** For example, in the case of the cardioid

$$r = a(1 - \cos \theta),$$

it is readily shown, from the results obtained in Art. 145, that the angle between the tangent and the radius vector is  $\frac{1}{2}\theta$ ; and therefore  $\phi = \frac{3}{2}\theta$ , and

$$\rho = \frac{ds}{d\phi} = \frac{4a}{3} \sin \frac{\phi}{3}.$$

To obtain the whole area between the curve and its evolute, the limits for  $\theta$  are 0 and  $2\pi$ ; hence the limits for  $\phi$  are 0 and  $3\pi$ . Therefore

$$A = \frac{1}{2} \int_0^{3\pi} \rho^2 d\phi = \frac{8a^2}{9} \int_0^{3\pi} \sin^2 \frac{\phi}{3} d\phi = \frac{4\pi a^2}{3}.$$

**159.** As another application of the general formula of Art. 156, let one end of a line of fixed length  $a$  be moved

along a given line in a horizontal plane, while a weight attached to the other extremity is drawn over the plane by the line, and is therefore always moving in the direction of the line itself. The line of fixed length in this case turns about the weight as a moving centre of rotation. Hence the area generated while the line turns through a given angle is the same as that of the corresponding sector of a circle whose radius is  $a$ .

The curve described by the weight is called a *tractrix*, and the line along which the other extremity is moved is the *directrix*. When the axis of  $x$  is the directrix, and the weight starts from the point  $(0, a)$ , the common tractrix is described; hence the area between this curve and the axes is  $\frac{1}{4}\pi a^2$ .

160. Again, in the generation of the cycloid, Diff. Calc., Art. 288 [Abridged Ed., Art. 156], the variable chord  $RP$  may be regarded as generating the area. The point  $R$  has a motion in the direction of the tangent  $RX$ ; the point  $P$  partakes of this motion, which is the motion of the centre  $C$ , and also has an equal motion, due to the rotation of the circle in the direction of the tangent to the circle at  $P$ . Since the tangents at  $P$  and  $R$  are equally inclined to  $PR$ , the motion of  $P$  in a direction perpendicular to  $PR$  is double the component, in this direction, of the motion of  $R$ . Therefore the centre of rotation of  $PR$  is beyond  $R$  at a distance from it equal to  $PR$ . Hence, denoting  $PRO$  by  $\phi$ ,

$$\rho_1 = PR = 2a \sin \phi, \qquad \rho_2 = 2PR = 4a \sin \phi.$$

Substituting in the formula of Art. 156, we have for the area of the cycloid, since  $PRO$  varies from 0 to  $\pi$ ,

$$A = 6a^2 \int_0^\pi \sin^2 \phi \, d\phi = 3\pi a^2.$$

### Sign of the Generated Area.

161. Let  $AB$  be the generating line, and  $C$  the centre of rotation. The expression,

$$dA = \frac{1}{2} (\rho_2^2 - \rho_1^2) d\phi, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the differential of the area, was obtained upon the supposition that  $A$  and  $B$  were on the same side of  $C$ . Then supposing  $\rho_2 > \rho_1$ , and that the line rotates in the positive direction, as in figure 29, *the differential of the area is positive*; and we notice that every point in the area generated is swept over by the line  $AB$ , *the left hand side as we face in the direction  $AB$  preceding*.



FIG. 29.

162. We shall now show that in every case, the formula requires that an area swept over with the left side preceding, shall be considered as positively generated, and one swept over in the opposite direction as negatively generated.

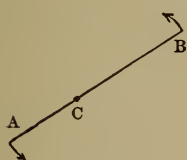


FIG. 30.

In the first place, if  $C$  is between  $A$  and  $B$  so that  $\rho_1$  is negative, as in figure 30,  $\rho_1^2$  is still positive, and formula (1) still gives the difference between the areas generated by  $CB$  and  $AC$ . Hence the latter area, which is now generated by a part of the line  $AB$ , must be regarded as generated negatively, but the *right hand side* as we face in the direction  $AB$  of this portion of the line is now preceding, which agrees with the rule given in Art. 161.

Again, if  $C$  is beyond  $B$ , the formula gives the difference of the generated areas; but since  $\rho_1^2$  is numerically greater than  $\rho_2^2$ , in this case,  $dA$  is negative, and the area generated by  $AB$  is the difference of the areas, and is negative by the rule.

Finally, if the direction of rotation be reversed,  $d\phi$  and therefore  $dA$  change sign, but the opposite side of each portion of the line becomes in this case the preceding side.

**163.** We may now put the expression for the area in another form. For

$$dA = \frac{1}{2}(\rho_2^2 - \rho_1^2) d\phi = (\rho_2 - \rho_1) \frac{\rho_2 + \rho_1}{2} d\phi;$$

whatever be the signs of  $\rho_2$  and  $\rho_1$ , the first factor is the length of  $AB$ , which we shall denote by  $l$ , and the second factor is the distance of the middle point of  $AB$  from the centre of rotation, which we shall denote by  $\rho_m$ . Hence, putting

$$\rho_2 - \rho_1 = l, \quad \text{and} \quad \frac{\rho_2 + \rho_1}{2} = \rho_m,$$

we have

$$A = \int l \rho_m d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since  $\rho_m d\phi$  is the differential of the motion of the middle point in a direction perpendicular to  $AB$ , this expression shows that the differential of the area is the product of this differential by the length of the generating line.

### *Areas generated by Lines whose Extremities describe Closed Circuits.*

**164.** Let us now suppose the generating line  $AB$  to move from a given position, and to return to the same position, each of the extremities  $A$  and  $B$  describing a closed curve in the positive direction, as indicated by the arrows in figure 31. It is readily seen that every point which is in the area described by  $B$ , and not in that described by  $A$ , will be swept over at least once by the line  $AB$ , the left side preceding, and if passed over more than once, there will be

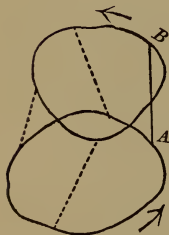


FIG. 31.



an excess of one passage, the left side preceding. Therefore the area within the curve described by  $B$ , and not within that described by  $A$ , will be generated positively. In like manner the area within the curve described by  $A$ , and not within that described by  $B$ , will be generated negatively. Furthermore, all points within both or neither of these curves are passed over, if at all, an equal number of times in each direction, so that the area common to the two curves and exterior to both disappears from the expression for the area generated by  $AB$ .

Hence it follows that, *regarding a closed area whose perimeter is described in the positive direction as positive, the area generated by a line returning to its original position is the difference of the areas described by its extremities.* This theorem is evidently true generally, if areas described in the opposite direction are regarded as negative.

### *Amsler's Planimeter.*

**165.** The theorem established in the preceding article may be used to demonstrate the correctness of the method by which an area is measured by means of the *Polar Planimeter*, invented by Professor Amsler, of Schaffhausen.

This instrument consists of two bars,  $OA$  and  $AB$ , Fig. 32, jointed together at  $A$ . The rod  $OA$  turns on a fixed pivot at  $O$ , while a tracer at  $B$  is carried in the positive direction completely around the perimeter of the area to be measured. At some point  $C$  of the bar  $AB$  a small wheel is fixed, having its axis parallel to  $AB$ , and its circumference resting upon the paper. When  $B$  is moved, this wheel has a sliding and a rolling motion; the latter motion is recorded by an attachment by means of which the number of turns and parts of a turn of the wheel are registered.

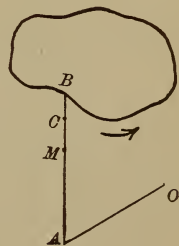


FIG. 32.

**166.** Let  $M$  be the middle point of  $AB$ , and let

$$OA = a, \quad AB = b, \quad MC = c.$$

Since  $b$  is constant, the area described by  $AB$  is by equation (2), Art. 163,

$$\text{Area } AB = b \int \rho_m d\phi. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Denoting the linear distance registered on the circumference of the wheel by  $s$ ,  $ds$  is the differential of the motion of the point  $C$ , in a direction perpendicular to  $AB$ , and since the distance of this point from the centre of rotation is  $\rho_m + c$ ,

$$ds = (\rho_m + c) d\phi:$$

substituting in (1) the value of  $\rho_m d\phi$ ,

$$\text{Area } AB = b \int ds - bc \int d\phi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

**167.** Two cases arise in the use of the instrument. When, as represented in Fig. 32,  $O$  is outside the area to be measured, the point  $A$  describes no area, and by the theorem of Art. 164, equation (2) represents simply the area described by  $B$ . In this case  $\phi$  returns to its original value, hence  $\int d\phi$  vanishes, and denoting the area to be measured by  $A$ , equation (2) becomes

$$A = bs. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the second case, when  $O$  is within the curve traced by  $B$ , the point  $A$  describes a circle whose area is  $\pi a^2$ , and the limit-

ing values of  $\phi$  differ by a complete revolution. Hence in this case equation (2) becomes

$$A - \pi a^2 = bs - 2\pi bc,$$

or 
$$A = bs + \pi(a^2 - 2bc).^* \quad . \quad . \quad . \quad . \quad (4)$$

In another form of the planimeter the point  $A$  moves in a straight line, and the same demonstration shows that the area is always equal to  $bs$ .

### Examples XIV.

1. The involute of a circle whose radius is  $a$  is drawn, and a tangent is drawn at the opposite end of the diameter which passes through the cusp; find the area between the tangent and the involute.

$$\frac{a^2 \pi (3 + \pi^2)}{3}.$$

2. Two radii vectores of a closed oval are drawn from a fixed point within, one of which is parallel to the tangent at the extremity of the other; if the parallelogram be completed, the area of the locus of its vertex is double the area of the given oval.

3. Show that the area of the locus of the middle point of the chord joining the extremities of the radii vectores in Ex. 2, is one half the area of the given oval.

\* The planimeter is usually so constructed that the positive direction of rotation is with the hands of a watch. The bar  $b$  is adjustable, but the distance  $AC$  is fixed so that  $c$  varies with  $b$ . Denoting  $AC$  by  $q$ , we have  $c = q - \frac{1}{2}b$ , and the constant to be added becomes  $C = \pi(a^2 - 2bq + b^2)$  in which  $a$  and  $q$  are fixed and  $b$  adjustable. In some instruments  $q$  is negative.

It is to be noticed that in the second case  $s$  may be negative; the area is then the numerical difference between the constant and  $bs$ .

4. Prove that the difference of the perimeters of two parallel ovals, whose distance is  $b$ , is  $2\pi b$ , and that the difference of their areas is the product of  $b$  and the half sum of their perimeters.

5. From a fixed point on the circumference of a circle whose radius is  $a$  a radius vector is drawn, and a distance  $b$  is measured from the circumference upon the radius vector produced; the extremity of  $b$  therefore describes a limaçon: show that the area generated by  $b$  when  $b > 2a$  is the area of the limaçon diminished by twice the area of the circle, and thence determine the area of the limaçon.

$$\pi(2a^2 + b^2).$$

6. Verify equation (4), Art. 167, when the tracer describes the circle whose radius is  $a + b$ .

7. Verify the value of the constant in equation (4), Art. 167, by determining the circle which may be described by the tracer without motion of the wheel.

8. If, in the motion of a crank and connecting rod (the line of motion of the piston passing through the centre of the crank), Amsler's recording wheel be attached to the connecting rod at the piston end, determine  $s$  geometrically, and verify by means of the area described by the other end of the rod.

9. The length of the crank in Ex. 8 being  $a$ , and that of the connecting rod  $b$ , find the area of the locus of a point on the connecting rod at a distance  $c$  from the piston end.

$$\frac{\pi a^2 c}{b}.$$

10. If a line  $AB$  of fixed length move in a plane, returning to its original position *without making a complete revolution*, denoting the areas of the curves described by its extremities by  $(A)$  and  $(B)$ , determine the area of the curve described by a point cutting  $AB$  in the ratio  $m : n$ .

$$\frac{n(A) + m(B)}{m + n}.$$

11. If the line in Ex. 10 return to its original position after *making a complete revolution*, prove *Holditch's Theorem*; namely, that the area of the curve described by a point at the distance  $c$  and  $c'$  from  $A$  and  $B$  is

$$\frac{c'(A) + c(B)}{c + c'} = \pi cc'.$$

12. Show by means of Ex. 11 that, if a chord of fixed length move around an oval, and a curve be described by a point at the distances  $c$  and  $c'$  from its ends, the area between the curves will be  $\pi cc'$ .

## XV.

### *Approximate Expressions for Areas and Volumes.*

168. When the equation of a curve is unknown, the area between the curve, the axis of  $x$ , and two ordinates may be approximately expressed in terms of the base and a limited number of ordinates, which are supposed to have been measured.

Let  $ABCDE$  be the area to be determined; denote the length of the base by  $2h$ ; and let the ordinates at the extremities and middle point of the base be measured and denoted by  $y_1, y_2$ , and  $y_3$ . Taking the base for the axis of  $x$ , and the middle point as origin, let it be assumed that the curve has an equation of the form

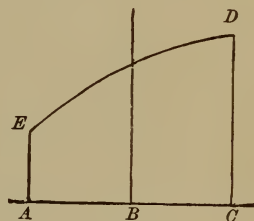


FIG. 33.

$$y = A + Bx + Cx^2 + Dx^3; \dots \dots \dots (1)$$

then the area required is

$$A = \int_{-h}^h y \, dx = Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} + \frac{Dx^4}{4} \Big|_{-h}^h = \frac{h}{3} (6A + 2Ch^2), \dots (2)$$

in which which  $A$  and  $C$  are unknown.

In order to express the area in terms of the measured ordinates, we have from equation (1),

$$y_1 = A + Bh + Ch^2 + Dh^3,$$

$$y_2 = A,$$

$$y_3 = A - Bh + Ch^2 - Dh^3;$$

whence we derive

$$y_1 + y_3 = 2A + 2Ch^2,$$

$$y_1 + 4y_2 + y_3 = 6A + 2Ch^2;$$

and substituting in (2),

$$A = \frac{h}{3} (y_1 + 4y_2 + y_3).$$

It will be noticed that this formula gives a perfectly accurate result when the curve is really a parabolic curve of the third or a lower degree.

**169.** If the base be divided into three equal intervals, each denoted by  $h$ , and the ordinates at the extremities and at the points of division measured, we have, by assuming the same equation,

$$A = \int_{-\frac{3h}{2}}^{\frac{3h}{2}} y \, dx = \frac{3h}{4} (4A + 3Ch^2) \quad \dots \quad (1)$$

From the equation of the curve,

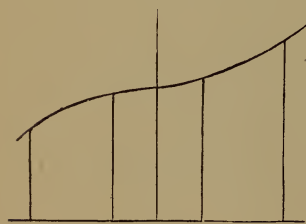


FIG. 34.

$$y_1 = A - \frac{3Bh}{2} + \frac{9Ch^2}{4} - \frac{27Dh^3}{8},$$

$$y_2 = A - \frac{Bh}{2} + \frac{Ch^2}{4} - \frac{Dh^3}{8},$$

$$y_3 = A + \frac{Bh}{2} + \frac{Ch^2}{4} + \frac{Dh^3}{8},$$

$$y_4 = A + \frac{3Bh}{2} + \frac{9Ch^2}{4} + \frac{27Dh^3}{8};$$

whence

$$y_1 + y_4 = 2A + \frac{9Ch^2}{2},$$

$$y_2 + y_3 = 2A + \frac{Ch^2}{2}.$$

From these equations we obtain

$$A = \frac{-y_1 + 9y_2 + 9y_3 - y_4}{16},$$

and

$$Ch^2 = \frac{y_1 - y_2 - y_3 + y_4}{4}.$$

Substituting in equation (1),

$$A = \frac{3h}{8} (y_1 + 3y_2 + 3y_3 + y_4).$$

### *Simpson's Rules.*

**170.** The formulæ derived in Articles 168 and 169, although they were first given by Cotes and Newton, are usually known as *Simpson's Rules*, the following extensions of the formulæ having been published in 1743, in his *Mathematical Dissertations*.

If the whole base be divided into an even number  $n$  of parts, each equal to  $h$ , and the ordinates at the points of division be numbered in order from end to end, then by applying the first formula to the areas between the alternate ordinates, we have

$$A = \frac{h}{3} (y_1 + 4y_2 + 2y_3 + 4y_4 \cdots + 4y_n + y_{n+1}).$$

That is to say, the area is equal to the product of the sum of the extreme ordinates, four times the sum of the even-num-



bered ordinates, and twice the sum of the remaining odd-numbered ordinates, multiplied by one third of the common interval.

Again, if the base be divided into a number of parts divisible by three, we have, by applying the formula derived in Art. 169, to the areas between the ordinates  $y_1 y_4$ ,  $y_4 y_7$ , and so on,

$$A = \frac{3h}{8} (y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 \cdots + 3y_n + y_{n+1}).$$

### *Cotes' Method of Approximation.*

171. The method employed in Articles 168 and 169 is known as *Cotes' Method*. It consists in assuming the given curve to be a parabolic curve of the highest order which can be made to pass through the extremities of a series of equidistant measured ordinates.

The equation of the parabolic curve of the  $n$ th order contains  $n + 1$  unknown constants; hence, in order to eliminate these constants from the expression for an area defined by the curve, it is in general necessary to have  $n + 1$  equations connecting them with the measured ordinates. Hence, if  $n$  denote the number of intervals between measured ordinates over which the curve extends, the curve will in general be of the  $n$ th degree.\*

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\* If  $H$  denotes the whole base, the first factor is always equivalent to  $H$  divided by the sum of the coefficients of the ordinates; for if all the ordinates are made equal, the expression must reduce to  $Hy_1$ . Thus, each of the rules for an approximate area, including those derived by repeated applications, as in Art. 170, may be regarded as giving an expression for the *mean ordinate*. The coefficients of the ordinates, according to Cotes' method, for all values of  $n$  up to  $n = 10$ , may be found in Bertrand's *Calcul Intégral*, pages 333 and 334. For example (using detached coefficients for brevity), we have, when  $n = 4$ ,

$$A = \frac{H}{90} [7, 32, 12, 32, 7];$$

and when  $n = 6$ ,

$$A = \frac{H}{840} [41, 216, 27, 272, 27, 216, 41].$$



172. For example, let it be required to determine the area between the ordinates  $y_1$  and  $y_2$ , in terms of the three equidistant ordinates  $y_1$ ,  $y_2$  and  $y_3$ , the common interval being  $h$ .

We must assume

$$y = A + Bx + Cx^2;$$

then taking the origin at the foot of  $y_1$ ,

$$A = \int_0^h y dx = h \left[ A + \frac{Bh}{2} + \frac{Ch^2}{3} \right],$$

from which  $A$ ,  $B$  and  $C$  must be eliminated by means of the equations

$$y_1 = A,$$

$$y_2 = A + Bh + Ch^2,$$

$$y_3 = A + 2Bh + 4Ch^2.$$

Solving these equations, we obtain

$$A = y_1,$$

$$Bh = \frac{-3y_1 + 4y_2 - y_3}{2},$$

$$Ch^2 = \frac{y_1 - 2y_2 + y_3}{2}.$$

If we make a slight modification in the ratios of these last coefficients by substituting for each the nearest multiple of 42, we have

$$A = \frac{H}{840} [42, 210, 42, 252, 42, 210, 42],$$

(the denominator remaining unchanged, since the sum of the coefficients is still 840), which reduces to

$$A = \frac{H}{20} [1, 5, 1, 6, 1, 5, 1].$$

This result is known as *Weddles' Rule* for six intervals. The value thus given to the mean ordinate is evidently a very close approximation to that resulting from Cotes' method, the difference being

$$\frac{1}{840} [y_1 + y_7 + 15(y_3 + y_5) - 6(y_2 + y_6) - 20y_4].$$

and substituting

$$A = \frac{h}{12} (5y_1 + 8y_2 - y_3).$$

**173.** It is, however, to be noticed, that when the ordinates are symmetrically situated with respect to the area, if  $n$  is *even*, the parabolic curve may be assumed of the  $(n + 1)$ th degree. For example, in Art. 168,  $n = 2$ , but the curve was assumed of the third degree. Inasmuch as  $A, B, C$  and  $D$  cannot all be expressed in terms of  $y_1, y_2$ , and  $y_3$ , we see that a variety of parabolic curves of the third degree can be passed through the extremities of the measured ordinates, but all of these curves have the same area.\*

### *Application to Solids.*

**174.** If  $y$  denotes the area of the section of a solid perpendicular to the axis of  $x$ , the volume of the solid is  $\int y \, dx$ , and

---

\* This circumstance indicates a probable advantage in making  $n$  an even number when repeated applications of the rules are made. Thus, in the case of six intervals, we can make three applications of Simpson's first rule, giving

$$A = \frac{H}{18} [1, 4, 2, 4, 2, 4, 1], \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or two of Simpson's second rule, giving

$$A = \frac{H}{16} [1, 3, 3, 2, 3, 3, 1]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the first case, we assume the curve to consist of three arcs of the third degree, meeting at the extremities of the ordinates  $y_3$  and  $y_5$ ; but, since each of these arcs contains an undetermined constant, we can assume them to have common tangents at the points of meeting. We have therefore a *smooth*, though not a continuous curve. In the second case, we have two arcs of the third degree containing no arbitrary constants, and therefore making an angle at the extremity of  $y_4$ . It is probable, therefore, that the smooth curve of the first case will in most cases form a better approximation than the broken curve of the second case.

In confirmation of this conclusion, it will be noticed that the ratios of the coefficients in equation (1) are nearer to those of Cotes' coefficients for  $n = 6$ , given in the preceding foot-note, than are those in equation (2).

therefore the approximate rules deduced in the preceding articles apply to solids as well as to areas. Indeed, they may be applied to the approximate computation of any integral, by putting  $y$  equal to the coefficient of  $dx$  under the integral sign.

The areas of the sections may of course be computed by the approximate rules.

### *Woolley's Rule.*

175. When the base of the solid is rectangular, and the ordinates of the sections necessary to the application of Simpson's first rule are measured, we may, instead of applying that rule, introduce the ordinates directly into the expression for the area in the following manner.

Taking the plane of the base for the plane of  $xy$ , and its centre for the origin, let the equation of the upper surface be assumed of the form

$$z = A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y + Ixy^2 + Jy^3.$$

Let  $2h$  and  $2k$  be the dimensions of the base, and denote the measured values of  $z$  as indicated in Fig. 35. The required volume is

$$V = \int_{-h}^h \int_{-k}^k z \, dy \, dx.$$

This double integral vanishes for every term containing an odd power of  $x$  or an odd power of  $y$ : hence

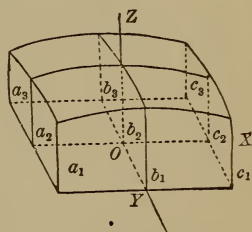


FIG. 35.

$$\begin{aligned} V &= 4Ahk + \frac{4Dh^3k}{3} + \frac{4Fhk^3}{3}, \\ &= \frac{hk}{3} [12A + 4Dh^2 + 4Fk^2]. \quad \dots \dots (I) \end{aligned}$$

By substituting the values of  $x$  and  $y$  in the equation of the surface, we readily obtain

$$b_2 = A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a_1 + a_3 + c_1 + c_3 = 4A + 4Dh^2 + 4Fk^2, \quad . \quad . \quad . \quad (3)$$

$$a_2 + c_2 + b_1 + b_3 = 4A + 2Dh^2 + 2Fk^2. \quad . \quad . \quad . \quad (4)$$

From these equations two very simple expressions for the volume may be derived ; for, employing (2) and (4), equation (1) becomes

$$V = \frac{2hk}{3} (a_2 + b_1 + 2b_2 + b_3 + c_2); \quad . \quad . \quad . \quad . \quad (4)$$

and employing (2) and (3),

$$V = \frac{hk}{3} (a_1 + a_3 + 8b_2 + c_1 + c_3). \quad . \quad . \quad . \quad . \quad (5)$$

Equation (4) is known as *Woolley's Rule* ; the ordinates employed are those at the middles of the sides and at the centre ; in (5), they are at the corners and at the centre.

### Examples XV.

1. Apply Simpson's Rule to the sphere, the hemisphere, and the cone, and explain why the results are perfectly accurate.

2. Apply Simpson's Second Rule to the larger segment of a sphere made by a plane bisecting at right angles a radius of the sphere.

$$\frac{9\pi a^3}{8}.$$

3. Find by Simpson's Rule the volume of a segment of a sphere,  $b$  and  $c$  being the radii of the bases, and  $h$  the altitude.

$$\frac{\pi h}{6} (3b^2 + 3c^2 + h^2).$$

4. Find by Simpson's Rule the volume of the frustum of a cone,  $b$  and  $c$  being the radii of the bases, and  $h$  the altitude.

$$\frac{\pi h}{3} (b^2 + bc + c^2).$$

5. Compute by Simpson's First and Second Rules, the value of  $\int_0^1 \frac{dx}{1+x}$ , the common interval being  $\frac{1}{12}$  in each case.

The first rule gives 0.6931487, and the second rule gives 0.6931505.

The correct value is obviously  $\log_e 2 = 0.6931472$ .

6. Find the volume considered in Art. 175, directly by Simpson's Rule, and show that the result is consistent with equations (4) and (5).

$$V = \frac{hk}{9} [a_1 + a_3 + c_1 + c_3 + 4(a_2 + b_1 + b_3 + c_2) + 16b_2].$$

7. Find, by elimination, from equations (4) and (5), Art. 175, a formula which can be used when the centre ordinate is unknown.

$$V = \frac{hk}{3} [4(a_2 + b_1 + b_3 + c_2) - (a_1 + a_3 + c_1 + c_3)].$$

## CHAPTER IV.

## MECHANICAL APPLICATIONS.

## XVI.

*Definitions.*

176. We shall give in this chapter a few of the applications of the Integral Calculus to mechanical questions.

The *mass* or quantity of matter contained in a body is proportional to its weight. When the masses of all parts of equal volume are equal, the body is said to be *homogeneous*. The factor by which it is necessary to multiply the unit of volume to produce the unit of mass is called the *density*, and usually denoted by  $\gamma$ .

In the following articles it will be assumed, when not otherwise stated, that the body is homogeneous, and that the density is equal to unity, so that the unit of mass is identical with the unit of volume. When the mass of an area is spoken of, it is regarded as a lamina of uniform thickness and density, and the unit of mass is taken to correspond with the unit of surface. In like manner the unit of mass for a line is taken as identical with the unit of length.

*Statical Moments.*

177. The *moment* of a force, with reference to a point, is the measure of the effectiveness of the force in producing motion about the point. It is shown in treatises on Mechanics, that this is *the product of the force and the perpendicular from the point upon the line of application of the force*.

The moment of the sum of a number of forces about a given point is the sum of the moments of the forces.

The *statical moment* of a body about a given point is the moment of its gravity; the force of gravity being supposed to act upon every part of the body, and in parallel lines.

**178.** In order to find the statical moment of a continuous body, we regard the body as generated geometrically in some convenient manner, and determine the corresponding differential of the moment.

In the case of a plane area, let the body be referred to rectangular axes, and let gravity be supposed to act in the direction of the axis of  $y$ . Then the abscissa of the point of application is the *arm* of the force when we consider the moment about the origin. Let us first suppose the area to be generated by the motion of the ordinate  $y$ . The differential of the area is then  $y \, dx$ . The corresponding element of the sum, of which the integral  $\int_a^b y \, dx$  is the limiting value, see Art. 92, is

$$y_r \Delta x, \dots \dots \dots (1)$$

in which  $y_r$  is the ordinate corresponding to *any* value of  $x$  intermediate between  $a + (r - 1)\Delta x$ , and  $a + r\Delta x$ . It is evident that the arm of the weight of the element (1) is such an intermediate value of  $x$ ; hence the moment of the element is

$$x_r y_r \Delta x. \dots \dots \dots (2)$$

The whole moment is therefore the limiting value of a sum of the form

$$\sum_a^b x_r y_r \Delta x.$$

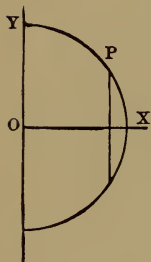
In other words, it is the integral

$$\int_a^b xy \, dx, \dots \dots \dots (3)$$

in which the differential of the moment is the product of the differential of the area and the arm of the force, which in this case is the same for every point of the element. In other words, *the moment of the differential is the differential of the moment.*

179. As an illustration, we find the moment of a semicircle (Fig. 36) about its centre. The area may be generated by the line  $2y$ , moving from  $x = 0$  to  $x = a$ . The equation of the circle being

$$x^2 + y^2 = a^2,$$



the differential of the area is

$$2 \sqrt{a^2 - x^2} dx.$$

The moment of this differential is

$$2 \sqrt{a^2 - x^2} x dx;$$

hence the whole moment is

$$2 \int_0^a \sqrt{a^2 - x^2} x dx = -\frac{2}{3} (a^2 - x^2)^{\frac{3}{2}} \Big|_0^a = \frac{2a^3}{3}.$$

### *Centres of Gravity.*

180. If a force equal to the whole weight of a body be applied with an arm properly determined, its moment may be made equivalent to the whole statical moment of the body. If the force is in the direction of the axis of  $y$ , as in Fig. 36, we have, denoting this arm by  $\bar{x}$ ,

$$\bar{x} \cdot \text{Area} = \text{Moment},$$

$$\bar{x} = \frac{\text{Moment}}{\text{Area}}.$$



In like manner, supposing the force to act in the direction of the axis of  $x$ , we may determine  $y$  for the same body.

It is shown in treatises on Mechanics that the point determined by the two coordinates  $\bar{x}$  and  $\bar{y}$ , is independent of the position of the coordinate axis. This point is called the *centre of gravity* of the area. The centre of gravity of a volume is defined in like manner.

**181.** The symmetry of the form of a body may determine one or more of the coordinates of its centre of gravity. Thus the centre of gravity of a circle or a sphere coincides with the geometrical centre, and the centre of gravity of a solid of revolution is on the axis of revolution. The centre of gravity of the semicircle in Fig. 36, is on the axis of  $x$ ; hence to determine its position we have only to find  $\bar{x}$ . Dividing the moment of the semicircle found in Art. 179 by the area  $\frac{1}{2}\pi a^2$ , we have

$$\bar{x} = \frac{4a}{3\pi}.$$

**182.** In finding the moment of the semicircle (Art. 179), we regarded the area as generated by the double ordinate  $2y$ , and the differential of the moment was found by multiplying the differential of the area by  $x$ , which is the arm of the force for every point of the generating line.

We may, however, derive the moment from the differential of area,

$$x dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since the area may be generated by the motion of the abscissa  $x$  from  $y = -a$  to  $y = a$ . But in this case to find the moment of the differential we must multiply it by the distance of its centre of gravity from the given axis. The centre of gravity of the line  $x$  is evidently its middle point, hence the required arm is  $\frac{1}{2}x$ . Therefore the differential of the moment is

$$\frac{x^2 dy}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and consequently the whole moment is

$$\frac{1}{2} \int_{-a}^a x^2 dy = \frac{1}{2} \int_{-a}^a (a^2 - y^2) dy = \frac{2a^3}{3}.$$

This result is identical with that derived in Art. 179.

### *Polar Formulæ.*

**183.** When polar formulæ are employed,  $r$  and  $\theta$  being coordinates of the curved boundary of the area, the element is  $\frac{1}{2}r^2 d\theta$ . Since this element is ultimately a triangle, we employ the well known property of triangles; that the centre of gravity is on a medial line at two-thirds the distance from the vertex to the base.

The coordinates of the centre of gravity of the element are, therefore,

$$\frac{2}{3}r \sin \theta \quad \text{and} \quad \frac{2}{3}r \cos \theta.$$

Hence we have the formula

$$\bar{x} = \frac{\int \frac{2}{3}r \cos \theta \cdot \frac{1}{2}r^2 d\theta}{\frac{1}{2} \int r^2 d\theta} = \frac{2}{3} \cdot \frac{\int r^3 \cos \theta d\theta}{\int r^2 d\theta},$$

and similarly

$$\bar{y} = \frac{2}{3} \cdot \frac{\int r^3 \sin \theta d\theta}{\int r^2 d\theta}$$

184. To illustrate, let us find the centre of gravity of the area enclosed by one loop of the lemniscata.

$$r^2 = a^2 \cos 2\theta.$$

$$\text{Whence } \bar{x} = \frac{2a}{3} \frac{\int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos \theta d\theta}{\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta} = \frac{4a}{3} \int_0^{\frac{\pi}{4}} (\cos 2\theta)^{\frac{3}{2}} \cos \theta d\theta.$$

$$\text{Put } \cos 2\theta = \cos^2 \phi, \quad \text{whence } \sin \phi = \sqrt{2} \sin \theta,$$

$$\text{and } \sqrt{2} \cos \theta d\theta = \cos \phi d\phi,$$

$$\therefore \bar{x} = \frac{2\sqrt{2}}{3} a \int_0^{\frac{\pi}{2}} \cos^4 \phi d\phi = \frac{2\sqrt{2}}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} a = \frac{\sqrt{2}}{8} \pi a.$$

### *Solids of Revolution.*

185. To find the centre of gravity of a solid of revolution, we take the axis of revolution as the axis of  $x$ , and the circle whose area is  $\pi y^2$  as the generating element. Replacing  $y$  in equation (3), Art. 178, by this expression, we have for the static moment

$$\pi \int_a^b xy^2 dx,$$

and for the abscissa of the centre of gravity

$$\bar{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$$

**186.** To illustrate, we find the centre of gravity of a spherical segment whose height is  $h$ . In this case, taking the origin at the vertex of the segment, and denoting the radius of the sphere by  $a$ , we have

$$y^2 = 2ax - x^2.$$

$$\text{Hence } \bar{x} = \frac{\int_0^h (2ax^2 - x^3) dx}{\int_0^h (2ax - x^2) dx} = \frac{\left[ \frac{2}{3}ax^3 - \frac{1}{4}x^4 \right]_0^h}{\left[ ax^2 - \frac{1}{3}x^3 \right]_0^h} = \frac{h}{4} \cdot \frac{8a - 3h}{3a - h}.$$

If the centre of gravity of the surface of the segment be required, since the differential of the surface is  $2\pi y ds$ , we easily obtain the general formula

$$\bar{x} = \frac{\int_0^h xy ds}{\int_0^h y ds},$$

and, in this case the curve being a circle,  $y ds = a dx$ ; hence, substituting, we have

$$\bar{x} = \frac{1}{2}h.$$

### *The Properties of Pappus.*

**187.** Let a solid be generated by the revolution of any plane figure about an exterior axis in its own plane. It is required to determine the volume and the surface thus generated.

It is evident that this solid may also be generated by a variable circular ring whose centre moves along the axis of revolution; denoting by  $y_1$  and  $y_2$  corresponding ordinates of

the outer and inner circles respectively, the area of this ring is  $\pi(y_1^2 - y_2^2)$ . Hence

$$V = \pi \int (y_1^2 - y_2^2) dx = 2\pi \int \frac{y_1 + y_2}{2} (y_1 - y_2) dx.$$

But this integral is the statical moment of the given figure, since  $y_1 - y_2$  is the generating element of its area, and  $\frac{y_1 + y_2}{2}$  is the corresponding arm. Denoting the area of the figure by  $A$ , we may therefore write

$$V = 2\pi \bar{y} A ;$$

that is, *the volume is the product of the area of the figure and the path described by its centre of gravity.*

The surface ( $S$ ) of this solid is, by Art. 149,

$$S = 2\pi \int y ds = 2\pi \bar{y} \int ds,$$

if  $\bar{y}$  denotes the ordinate of the centre of gravity of the arc  $s$ .

Hence we have  $S = 2\pi \bar{y} \cdot \text{arc} ;$

that is, *the surface is the product of the length of the arc into the path described by the centre of gravity.*

These theorems are frequently called the properties of Guldinus; they are, however, due to Pappus, who published them 1588.

It is obvious that both theorems are true for any part of a revolution of the generating figure.

## Examples XVI.

1. Find the centre of gravity of the area enclosed between the parabola  $y^2 = 4mx$  and the double ordinate corresponding to the abscissa  $a$ .

$$\bar{x} = \frac{3a}{5}.$$

2. Find the centre of gravity of the area between the semi-cubical parabola  $ay^2 = x^3$  and the double ordinate which corresponds to the abscissa  $a$ .

$$\bar{x} = \frac{5a}{7}.$$

3. Find the ordinate of the centre of gravity of the area between the axis of  $x$  and the sinusoid  $y = \sin x$ , the limits being  $x = 0$  and  $x = \pi$ .

$$\bar{y} = \frac{1}{8}\pi.$$

4. Find the coordinates of the centre of gravity of the area between the axes and the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$

$$\bar{x} = \frac{a}{5}, \text{ and } \bar{y} = \frac{b}{5}.$$

5. Find the centre of gravity of the area between the *cisoid*  $y^2 (a - x) = x^3$  and its asymptote.

*Solution* :—

Denoting the statical moment by  $M$  and the area by  $A$ ,

$$\begin{aligned} M &= \int_0^a \frac{x^{\frac{5}{2}} dx}{(a-x)^{\frac{1}{2}}} = -2x^{\frac{5}{2}}(a-x)^{\frac{1}{2}} \Big|_0^a + 5 \int_0^a x^{\frac{3}{2}}(a-x)^{\frac{1}{2}} dx \\ &= 5a \cdot A - 5M; \end{aligned}$$

$$\therefore M = \frac{5a}{6} A,$$

hence

$$\bar{x} = \frac{5a}{6}.$$

6. Find the centre of gravity of the area between the parabola  $v^2 = 4ax$  and the straight line  $y = mx$ .

$$\bar{x} = \frac{8a}{5m^2}, \text{ and } \bar{y} = \frac{2a}{m}.$$

7. Find the centre of gravity of the segment of an ellipse cut off by a quadrantal chord.

$$\bar{x} = \frac{2}{3} \cdot \frac{a}{\pi - 2}, \text{ and } \bar{y} = \frac{2}{3} \cdot \frac{b}{\pi - 2}.$$

8. Given the cycloid,

$$y = a(1 - \cos \psi),$$

$$x = a(\psi - \sin \psi),$$

find the distance of the centre of gravity of the area from the base.

$$\bar{y} = \frac{5a}{6}.$$

9. Find the centre of gravity of the area enclosed between the positive directions of the coordinate axes and the four-cusped hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Put  $x = a \cos^3 \theta$ , and  $y = a \sin^3 \theta$ .

$$\bar{x} = \bar{y} = \frac{256a}{315\pi}.$$

10. Find the centre of gravity of the area enclosed by the cardioid

$$r = a(1 - \cos \theta).$$

$$\bar{x} = -\frac{5a}{6}.$$

11. Find the centre of gravity of the sector of a circle whose radius is  $a$ , the angle of the sector being  $2\alpha$ .

Use the method of Art. 183.

$$\bar{x} = \frac{2}{3} \frac{a \sin \alpha}{\alpha}$$

12. Find the centre of gravity of the segment of a circle, the angle subtended being  $2\alpha$  and the radius of the circle  $a$ .

*Solution :—*

$$\bar{x} = \frac{2 \int_{a \cos \alpha}^a (a^2 - x^2)^{\frac{1}{2}} x \, dx}{\text{Area}} = \frac{2a^3 \sin^3 \alpha}{3 \text{ Area}} = \frac{\text{Chord}^3}{12 \text{ Area}}.$$

13. Find the centre of gravity of a circular ring, the radii being  $a$  and  $a_1$ , and the angle subtended  $2\alpha$ .

$$\bar{x} = \frac{2}{3} \cdot \frac{a^3 - a_1^3}{a^2 - a_1^2} \cdot \frac{\sin \alpha}{\alpha}.$$

14. Find the centre of gravity of a circular arc, whose length is  $2s$ .

*Solution :—*

We have in this case, taking the origin at the centre and the axis of  $x$  bisecting the arc,

$$\bar{x} = \frac{\int_{-s}^s x \, ds}{\int_{-s}^s ds}.$$

Put  $x = a \cos \theta$ , then  $ds = a \, d\theta$ , and denoting by  $\alpha$  the angle subtended by  $s$ , we have

$$\bar{x} = \frac{a^2 \int_{-\alpha}^{\alpha} \cos \theta \, d\theta}{2s} = \frac{a \sin \alpha}{\alpha} = \frac{c}{\alpha},$$

$2c$  being the chord.



15. Find the coordinates of the centre of gravity of arc of the semi-cycloid whose equations, referred to the vertex, are

$$x = a(1 - \cos \psi), \quad \text{and} \quad y = a(\psi + \sin \psi).$$

$$\bar{x} = \frac{2a}{3}, \text{ and } \bar{y} = \left(\pi - \frac{4}{3}\right)a.$$

16. Find the centre of gravity of the arc between two successive cusps of the four-cusped hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

$$\bar{x} = \bar{y} = \frac{2a}{5}.$$

17. Find the position of the centre of gravity of the arc of the semi-cardioid

$$r = a(1 - \cos \theta).$$

$$\bar{x} = -\frac{4a}{5}, \text{ and } \bar{y} = \frac{4a}{5}.$$

18. A semi-ellipsoid is formed by the revolution of a semi-ellipse about its major axis; find the distance of the centre of gravity of the solid from the centre of the ellipse.

$$\bar{x} = \frac{3a}{8}.$$

19. Find the centre of gravity of a frustum of a paraboloid of revolution having a single base,  $h$  denoting the height of the frustum.

$$\bar{x} = \frac{2h}{3}.$$

20. A paraboloid and a cone have a common base and vertices at the same point; find the centre of gravity of the solid enclosed between them.

The centre of gravity is the middle point of the axis.

21. Find the centre of gravity of a hyperboloid whose height is  $h$ , the generating curve being

$$y^2 = m(2ax + x^2).$$

$$\bar{x} = \frac{h}{4} \cdot \frac{8a + 3h}{3a + h}.$$

22. Find the centre of gravity of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

The height of the cone being denoted by  $h$ , and the radius of the circle by  $a$ , we have

$$\bar{x} = \frac{3}{8}(a + h).$$

23. Find the centre of gravity of the solid formed by the revolution about the axis of  $x$  of the curve

$$a^2y = ax^2 - x^3,$$

between the limits 0 and  $a$ .

$$\bar{x} = \frac{5a}{8}.$$

24. A solid is formed by revolving about its axis the cardioid

$$r = a(1 - \cos \theta);$$

find the distance of the cusp from the centre of gravity.

$$\bar{x} = \frac{4a}{5}.$$

25. Determine the position of the centre of gravity of the volume included between the surfaces generated by revolving about the axis of  $x$  the two parabolas

$$y^2 = mx, \quad \text{and} \quad y^2 = m'(a - x).$$

$$\bar{x} = \frac{a}{3} \cdot \frac{m + 2m'}{m + m'}.$$

26. Find the centre of gravity of a rifle bullet consisting of a cylinder two calibers in length, and a paraboloid one and a half calibers in length having a common base, the opposite end of the cylinder containing a conical cavity one caliber in depth with a base equal in size to that of the cylinder.

The distance of the centre of gravity from the base of the bullet is  $1\frac{3}{5}$  calibers.

27. A solid formed by the revolution of a circular segment about its chord is cut in halves by a plane perpendicular to the chord; determine the centre of gravity of one of the halves. This solid is called an *ogival*.

Denoting by  $2\alpha$  the angle subtended by the chord, and by  $a$  the radius of the circle, the distance of the centre of gravity from the base is

$$\bar{x} = \frac{a}{16} \cdot \frac{44 \sin^2 \alpha + \sin^2 2\alpha + 32 (\cos 2\alpha - \cos \alpha)}{\sin \alpha (2 + \cos^2 \alpha) - 3\alpha \cos \alpha}.$$

28. Find the centre of gravity of the surface of the paraboloid formed by the revolution about the axis of  $x$  of the parabola

$$y^2 = 4mx,$$

$a$  denoting the height of the paraboloid.

$$\bar{x} = \frac{1}{5} \cdot \frac{(3a - 2m)(a + m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(a + m)^{\frac{3}{2}} - m^{\frac{3}{2}}}.$$

29. Find the centre of gravity of the surface generated by the revolution of a semi-cycloid about its axis, the equations of the curve being

$$x = a(1 - \cos \psi), \quad \text{and} \quad y = a(\psi + \sin \psi).$$

$$\bar{x} = \frac{2a}{15} \cdot \frac{15\pi - 8}{3\pi - 4}.$$

30. Find the centre of gravity of the surface generated by the revolution about its axis of one of the loops of the lemniscata

$$r^2 = a^2 \cos 2\theta.$$

$$\bar{x} = \frac{2 + \sqrt{2}}{6} a.$$

31. A cardioid revolves about its axis; find the centre of gravity of the surface generated, the equation of the cardioid being

$$r = a(1 - \cos \theta).$$

$$\bar{x} = -\frac{50a}{63}.$$

32. A ring is generated by the revolution of a circle about an axis in its own plane;  $c$  being the distance of the centre of the circle from the axis, and  $a$  the radius, determine the volume and surface generated.

$$V = 2\pi^2 ca^2, \text{ and } S = 4\pi^2 ca.$$

33. A triangle revolves about an axis in its plane;  $a_1$ ,  $a_2$ , and  $a_3$ , denoting the distances of its vertices from the axis, determine the volume generated.

$$V = \frac{2\pi A}{3} (a_1 + a_2 + a_3).$$

34. Find the volume of a frustum of a cone, the radii of the bases being  $a_1$  and  $a_2$ , and the height  $h$ .

$$V = \frac{\pi h}{3} (a_1^2 + a_1 a_2 + a_2^2).$$

35. Find the volume and surface generated by the revolution of a cycloid about its base.

$$V = 5\pi^2 a^3, \text{ and } S = \frac{64\pi a^2}{3}.$$

## XVII.

*Moments of Inertia.*

188. When a body rotates about a fixed axis, the velocity of a particle at a distance  $r$  from the axis is

$$r \frac{d\omega}{dt},$$

in which  $\omega$  is the angle of rotation. The force which acting for a unit of time would produce this motion in a mass  $m$  is measured by the momentum

$$mr \frac{d\omega}{dt}.$$

The moment of this force about the axis is therefore

$$mr^2 \frac{d\omega}{dt}.$$

The sum of these moments for all the parts of a rigid system is

$$\frac{d\omega}{dt} \sum r^2 \Delta m,$$

since the angular velocity,  $\frac{d\omega}{dt}$ , is constant. In the case of a continuous body this expression becomes

$$\frac{d\omega}{dt} \int r^2 dm,$$

in which  $dm$  is the differential of the mass. The factor

$$\int r^2 dm,$$

which depends upon the shape of the body, is called its *moment of inertia*, and is denoted by  $I$ .

189. When the body is homogeneous,  $dm$  is to be taken equal to the differential of the line, area, or volume, as the case may be. For example, in finding the moment of inertia of a straight line whose length is  $2a$ , about an axis bisecting it at right angles, we let  $x$  denote the distance of any point from the axis; then  $dm = dx$ , hence we have

$$I = \int_{-a}^a x^2 dx = \frac{2a^3}{3} = \frac{(2a)^3}{12}.$$

Again, in finding the moment of inertia of the semi-circle in figure 36, about the axis of  $y$ , let  $dm = 2y dx$ ; then, since every point of the generating line is at the distance  $x$  from the axis, the moment of inertia is

$$I = 2 \int_0^a y x^2 dx = 2 \int_0^a \sqrt{a^2 - x^2} x^2 dx.$$

Putting  $x = a \sin \theta$ , we have

$$I = 2a^4 \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin^2 \theta d\theta = \frac{\pi a^4}{8}.$$

### *The Radius of Gyration.*

190. If the whole mass of the body were situated at the distance  $k$  from the axis, its moment of inertia would be  $k^2 m$ . Now, if  $k$  is so determined that *this moment* shall be equal to the actual moment of inertia of the body, the value of  $k$  is *the radius of gyration* of the body with reference to the given axis. Hence

$$k^2 = \frac{\text{Moment of inertia}}{\text{Mass}}.$$

Thus, for the radius of gyration of the line  $2a$ , whose moment of inertia is found in the preceding article, we have

$$k^2 = \frac{a^2}{3}, \quad \text{or} \quad k = \frac{a}{\sqrt{3}};$$

and for the radius of gyration of the semi-circle, whose area is  $\frac{1}{2}\pi a^2$ ,

$$k^2 = \frac{a^2}{4}, \quad \text{or} \quad k = \frac{a}{2}.$$

It is evident that this expression is also the radius of gyration of the whole circle about a diameter, for the moment of inertia of the circle is evidently double that of the semi-circle, and its area is also double that of the semi-circle.

**191.** It is sometimes convenient to use modes of generating the area or volume, other than those involving rectangular coordinates. For example, let it be required to find the radius of gyration of a circle whose radius is  $a$ , about an axis passing through its centre and perpendicular to its plane. This circle may be generated by the circumference of a variable circle whose radius is  $r$ , while  $r$  passes from 0 to  $a$ . The differential of the area is then  $2\pi r dr$ , and the moment is

$$I = 2\pi \int_0^a r^3 dr = \frac{\pi a^4}{2}.$$

Dividing by the area of the circle, we have

$$k^2 = \frac{a^2}{2}.$$

**192.** Again, to find the radius of gyration of a sphere whose radius is  $a$  about a diameter. In order that all points of the elements shall be at the same distance from the axis

we regard the sphere as generated by the surface of a cylinder whose radius is  $x$ , and whose altitude is  $2y$ . The surface of this cylinder is therefore  $4\pi xy$ . The differential of the volume is  $4\pi xy dx$ , and the moment of inertia is

$$I = 4\pi \int x^3 y dx = 4\pi \int \sqrt{(a^2 - x^2)} x^3 dx.$$

Putting  $x = a \sin \theta$ ,

$$I = 4\pi a^5 \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta d\theta = \frac{8\pi a^5}{15}.$$

Dividing by  $\frac{4\pi a^3}{3}$ , the volume of the sphere, we have

$$k^2 = \frac{2a^2}{5}.$$

### *Radii of Gyration about Parallel Axes.*

**193.** *The moment of inertia of a body about any axis exceeds its moment of inertia about a parallel axis passing through the centre of gravity, by the product of the mass and the square of the distance between the axes.*

Let  $h$  be the distance between the axes. Pass a plane through the element  $dm$  perpendicular to the axes, and let  $r$  and  $r_1$  be the distances of the element from the axes. Then,  $r$ ,  $r_1$ , and  $h$  form a triangle; let  $\theta$  be the angle at the axis passing through the centre of gravity, then

$$r^2 = r_1^2 + h^2 - 2r_1 h \cos \theta. \quad \dots \dots (1)$$



The moment of inertia is therefore

$$\int r^2 dm = \int r_1^2 dm + h^2 m - 2h \int r_1 \cos \theta dm. \quad (2)$$

Now  $r_1$  and  $\theta$  are the polar coordinates of  $dm$ , in the plane which is passed through the element; hence the last integral in equation (2) is equivalent to

$$- 2h \int x dm.$$

But  $\int x dm$  is the statical moment of the body about the axis passing through the centre of gravity. Now from the definition of the centre of gravity, this moment is zero; hence, equation (2) reduces to

$$\int r^2 dm = \int r_1^2 dm + h^2 m. \quad (3)$$

Introducing the radii of gyration, we have also

$$k^2 = k_1^2 + h^2. \quad (4)$$

**194.** As an application of this result, we shall now find the moment of inertia of a cone whose height is  $h$ , and the radius of whose base is  $a$ , about an axis passing through its vertex perpendicular to its geometrical axis. Taking the origin at the vertex of the cone, the axis of  $x$  coincident with the geometrical axis, and a circle perpendicular to this axis as the generating element, we have for the area of this element  $\pi y^2$ , and for its radius of gyration about a diameter parallel to the given axis,  $\frac{y}{2}$  (Art. 190).

The distance between these axes being  $x$ , the proposition proved in the preceding article gives an expression for the radius of gyration of the element about the given axis; viz.,  $x^2 + \frac{y^2}{4}$ . Replacing  $r^2$ , in the general expression for  $I$  (Art. 188), by this expression, and substituting for  $dm$  the differential  $\pi y^2 dx$ , we have

$$I = \pi \int \left( x^2 + \frac{y^2}{4} \right) y^2 dx,$$

in which  $y = \frac{ax}{h}$ . Therefore

$$I = \frac{\pi a^2}{h^2} \int_0^h \left( 1 + \frac{a^2}{4h^2} \right) x^4 dx = \frac{\pi a^2 h^3}{5} \left( 1 + \frac{a^2}{4h^2} \right),$$

and since

$$V = \frac{\pi a^2 h}{3},$$

$$k^2 = \frac{3}{20} \left( a^2 + 4h^2 \right).$$

To find the square of the radius of gyration about a parallel axis through the centre of gravity, we have

$$\begin{aligned} k_o^2 &= \frac{3}{20} \left( a^2 + 4h^2 \right) - \left( \frac{3h}{4} \right)^2 \\ &= \frac{3}{80} \left( 4a^2 + h^2 \right). \end{aligned}$$

To find the moment of inertia of a right cone about its geometrical axis we employ the same generating element as before; but in this case the square of the radius of gyration is  $\frac{y^2}{2}$ . Hence

$$I = \frac{\pi}{2} \int y^4 dx = \frac{\pi a^4}{2h^4} \int_0^h x^4 dx;$$

therefore

$$I = \frac{\pi a^4 h}{10}, \quad \text{whence} \quad k^2 = \frac{3a^2}{10}.$$

### *Polar Moments of Inertia.*

195. In the case of a plane area, when the axis of rotation passes through the origin, we have

$$r^2 = x^2 + y^2, \quad \text{hence} \quad \int r^2 dm = \int (x^2 + y^2) dm,$$

therefore

$$I = \int x^2 dm + \int y^2 dm;$$

that is, *the sum of the moments of inertia of a plane area about two axes in its own plane at right angles to each other is equal to the moment of inertia about an axis through the origin perpendicular to the plane.*  $I$  in the above equation is called *the polar moment of inertia.*

In the case of the circle, since the moment is the same about every diameter, the polar moment is twice the moment about a diameter; that is, denoting the former by  $I_p$  and the latter by  $I_a$ , we have

$$I_p = 2 I_a = \frac{\pi a^4}{2}.$$

See Art. 191.

### Examples XVII.

1. Find the radius of gyration of a circular arc ( $2s$ ) about a radius passing through its vertex.

*Solution :—*

Taking the origin at the centre, and the axis of  $x$  bisecting the arc, and denoting by  $2\alpha$  the angle subtended by  $2s$ , we have

$$mk^2 = \int_{-s}^s y^2 ds = a^3 \int_{-\alpha}^{\alpha} \sin^2 \theta d\theta.$$

$$m = 2a\alpha \quad \therefore \quad k^2 = \frac{a^2}{2} \left( 1 - \frac{\sin 2\alpha}{2\alpha} \right)$$

2. Find the radius of gyration of the same arc about the axis of  $y$ , and thence about a perpendicular axis through the centre of the circle.

$$k = a.$$

3. Find the radius of gyration of the same arc about an axis through its vertex perpendicular to the plane of the circle.

See Ex. XVI., 14, and denote by  $c$  the subtending chord.

$$k^2 = 2a^2 \left( 1 - \frac{c}{2s} \right).$$

4. Find the moment of inertia of the chord of a circular arc, in terms of the diameter parallel to it, and its angular distance from this diameter.

See Arts. 189 and 193.

$$I = \frac{d^3}{24} (3 \cos \alpha - \cos 3\alpha).$$

5. Find the radius of gyration of an ellipse about an axis through its centre perpendicular to its plane.

Find the radius of gyration about the major axis and about the minor axis, and apply Art. 195.

$$k^2 = \frac{1}{4}(a^2 + b^2).$$

6. Find the radius of gyration of an isosceles triangle about a perpendicular let fall from its vertex upon the base ( $2b$ ).

$$k^2 = \frac{b^2}{6}.$$

7. Find the radius of gyration about the axis of the curve, of the area enclosed by the two loops of the lemniscata

$$r^2 = a^2 \cos 2\theta.$$

$$k^2 = \frac{a^2}{48} (3\pi - 8).$$

8. Find the radius of gyration of a right triangle, whose sides are  $a$  and  $b$ , about an axis through its centre of gravity perpendicular to its plane

$$k^2 = \frac{a^2 + b^2}{18}.$$

9. Find the radius of gyration of a portion of a parabola bounded by a double ordinate perpendicular to the axis, about a perpendicular to its plane passing through its vertex.

$$k^2 = \frac{3}{7} x^2 + \frac{1}{5} y^2.$$

10. Find the radius of gyration of a cylinder about a perpendicular that bisects its geometrical axis,  $2l$  being the length of the cylinder, and  $a$  the radius of its base.

$$k^2 = \frac{a^2}{4} + \frac{l^2}{3}.$$

11. Find the radius of gyration of a concentric spherical shell about a tangent to the external sphere, the radii being  $a$  and  $b$ .

$$k^2 = \frac{7a^5 - 5a^2b^3 - 2b^5}{5(a^3 - b^3)}.$$

12. Find the radius of gyration of a paraboloid of revolution about its axis, in terms of the radius ( $b$ ) of the base.

$$k^2 = \frac{b^2}{3}.$$

13. Find the moment of inertia of an ellipsoid about one of its principal axes.

$$I = \frac{4\pi abc}{15} (b^2 + c^2).$$

14. Find the radius of gyration of a symmetrical double convex lens about its axis,  $a$  being the radius of the circular intersection of the two surfaces, and  $b$  the semi-axis.

$$k^2 = \frac{b^4 + 5a^2b^2 + 10a^4}{10(b^2 + 3a^2)}.$$

15. Find the radius of gyration of the same lens about a diameter to the circle in which the spherical surfaces intersect.

$$k^2 = \frac{10a^4 + 15a^2b^2 + 7b^4}{20(b^2 + 3a^2)}.$$







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